Homework #3, Math 266, Summer 2010

1. We give an exhaustive proof—just check the entire domain. For $n = 1$ we have $1^2 + 1 = 2 \geq 2 = 2^1$. For $n = 2$ we have $2^2 + 1 = 5 \geq 4 = 2^2$. For $n = 3$ we have $3^2 + 1 = 10 \geq 8 = 2^3$. For $n = 4$ we have $4^2 + 1 = 17 \geq 16 = 2^4$. Notice that for $n \geq 5$, the inequality is no longer true.

5. There are several cases to consider. If x and y are both nonnegative, then $|x|+|y| = x+y = |x+y|$. Similarly, if both are negative, then $|x| + |y| = (-x) + (-y) = -(x + y) = |x + y|$, since $x + y$ is negative in this case. The complication (and strict inequality) comes if one of the variables is nonnegative and the other is negative. By the symmetry of the roles of x and y here (strictly speaking, by the commutativity of addition), we can assume without loss of generality that it is x that is nonnegative and y that is negative. So we have $x \geq 0$ and $y < 0$.

Now there are two subcases to consider within this case, depending on the relative sizes of the nonnegative numbers x and $-y$. First suppose that $x \geq -y$. Then $x + y \geq 0$. Therefore $|x + y| = x + y$, and this quantity is a nonnegative number smaller than x (since y is negative). On the other hand $|x| + |y| = x + |y|$ is a positive number bigger than x. Therefore we have $|x + y| < x < |x| + |y|$, as desired.

Finally, consider the possibility that $x < -y$. Then $|x + y| = -(x + y) = (-x) + (-y)$ is a positive number less than or equal to $-y$ (since $-x$ is nonpositive). On the other hand $|x| + |y| = |x| + (-y)$ is a positive number greater than or equal to $-y$. Therefore we have $|x + y| \leq -y \leq |x| + |y|$, as desired.

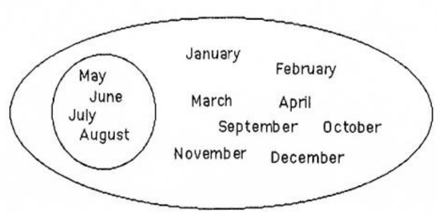
12. An assertion like this one is implicitly universally quantified—it means that *for all* rational numbers a and b, a^b is rational. To disprove such a statement it suffices to provide one counterexample. Take $a = 2$ and $b = 1/2$. Then $a^b = 2^{1/2} = \sqrt{2}$, and we know from Example 10 in Section 1.6 that $\sqrt{2}$ is not rational.

28. If $|y| \geq 2$, then $2x^2 + 5y^2 \geq 2x^2 + 20 \geq 20$, so the only possible values of y to try are 0 and ± 1. In the former case we would be looking for solutions to $2x^2 = 14$ and in the latter case to $2x^2 = 9$. Clearly there are no integer solutions to these equations, so there are no solutions to the original equation.

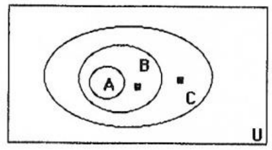
32. One proof that $\sqrt[3]{2}$ is irrational is similar to the proof that $\sqrt{2}$ is irrational, given in Example 10 in Section 1.6. It is a proof by contradiction. Suppose that $2^{1/3}$ (or $\sqrt[3]{2}$, which is the same thing) is the rational number p/q, where p and q are positive integers with no common factors (the fraction is in lowest terms). Cubing, we see that $2 = p^3/q^3$, or, equivalently, $p^3 = 2q^3$. Thus p^3 is even. Since the product of odd numbers is odd, this means that p is even, so we can write $p = 2s$. Substituting into the equation $p^3 = 2q^3$, we obtain $8s^3 = 2q^3$, which simplifies to $4s^3 = q^3$.

Now we play the same game with q. Since q^3 is even, q must be even. We have now concluded that p and q are both even, that is, that 2 is a common divisor of p and q. This contradicts the choice of p/q to be in lowest terms. Therefore our original assumption—that $\sqrt[3]{2}$ is rational—is in error, so we have proved that $\sqrt[3]{2}$ is irrational.

11. The four months whose names don't contain the letter R form a subset of the set of twelve months, as shown here.



13. We put the subsets inside the supersets. We also put dots in certain regions to indicate that those regions are not empty (required by the fact that these are proper subset relations). Thus the answer is as shown.



25. This is the set of triples (a, b, c), where a is an airline and b and c are cities. For example, (TWA, Rochester Hills Michigan, Middletown New Jersey) is an element of this Cartesian product. A useful subset of this set is the set of triples (a, b, c) for which a flies between b and c. For example, (Northwest, Detroit, New York) is in this subset, but the triple mentioned earlier is not.

31. The only difference between $A \times B \times C$ and $(A \times B) \times C$ is parentheses, so for all practical purposes one can think of them as essentially the same thing. By Definition 10, the elements of $A \times B \times C$ consist of 3-tuples (a, b, c), where $a \in A$, $b \in B$, and $c \in C$. By Definition 9, the elements of $(A \times B) \times C$ consist of ordered pairs (p, c), where $p \in A \times B$ and $c \in C$, so the typical element of $(A \times B) \times C$ looks like $((a, b), c)$. A 3-tuple is a different creature from a 2-tuple, even if the 3-tuple and the 2-tuple in this case convey exactly the same information. To be more precise, there is a natural one-to-one correspondence between $A \times B \times C$ and $(A \times B) \times C$ given by $(a, b, c) \leftrightarrow ((a, b), c)$.

35. In each case we want the set of all values of x in the domain (the set of integers) that satisfy the given equation or inequality.

a) The only integers whose squares are less than 3 are the integers whose absolute values are less than 2. So the truth set is $\{x \in \mathbf{Z} \mid x^2 < 3\} = \{-1, 0, 1\}$.

b) All negative integers satisfy this inequality, as do all nonnegative integers other than 0 and 1. So the truth set is $\{x \in \mathbf{Z} \mid x^2 > x\} = \mathbf{Z} - \{0, 1\} = \{\ldots, -2, -1, 2, 3, 4, \ldots\}$.

c) The only real number satisfying this equation is $x = -1/2$. Because this value is not in our domain, the truth set is empty: $\{x \in \mathbf{Z} \mid 2x + 1 = 0\} = \emptyset$.

1. **a)** the set of students who live within one mile of school and walk to class (only students who do both of these things are in the intersection)

b) the set of students who either live within one mile of school or walk to class (or, it goes without saying, both)

c) the set of students who live within one mile of school but do not walk to class

d) the set of students who live more than a mile from school but nevertheless walk to class

3. **a)** We include all numbers that are in one or both of the sets, obtaining $\{0, 1, 2, 3, 4, 5, 6\}$.

b) There is only one number in both of these sets, so the answer is $\{3\}$.

c) The set of numbers in A but not in B is $\{1, 2, 4, 5\}$.

d) The set of numbers in B but not in A is $\{0, 6\}$.

7. These identities are straightforward applications of the definitions and are most easily stated using set-builder notation. Recall that $\mathbf{T}$ means the proposition that is always true, and $\mathbf{F}$ means the proposition that is always false.

a) $A \cup U = \{\, x \mid x \in A \vee x \in U \,\} = \{\, x \mid x \in A \vee \mathbf{T} \,\} = \{\, x \mid \mathbf{T} \,\} = U$

b) $A \cap \emptyset = \{\, x \mid x \in A \wedge x \in \emptyset \,\} = \{\, x \mid x \in A \wedge \mathbf{F} \,\} = \{\, x \mid \mathbf{F} \,\} = \emptyset$

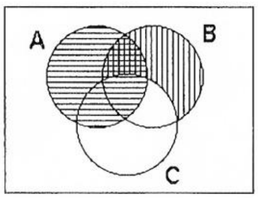
19. This is clear, since both of these sets are precisely $\{\, x \mid x \in A \wedge x \notin B \,\}$.

27. a) In the figure we have shaded the A set with horizontal bars (including the double-shaded portion, which includes both horizontal and vertical bars), and we have shaded the set $B - C$ with vertical bars (that portion inside B but outside C. The intersection is where these overlap—the double-shaded portion (shaped like an arrowhead).

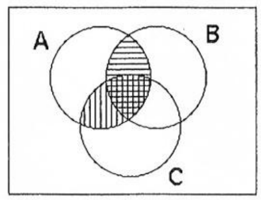
b) In the figure we have shaded the set $A \cap B$ with horizontal bars (including the double-shaded portion, which includes both horizontal and vertical bars), and we have shaded the set $A \cap C$ with vertical bars. The union is the entire region that has any shading at all (shaped like a tilted mustache).

c) In the figure we have shaded the set $A \cap \overline{B}$ with horizontal bars (including the double-shaded portion, which includes both horizontal and vertical bars), and we have shaded the set $A \cap \overline{C}$ with vertical bars. The union is the entire region that has any shading at all (everything inside A except the triangular middle portion where all three sets overlap) portion (shaped like an arrowhead).

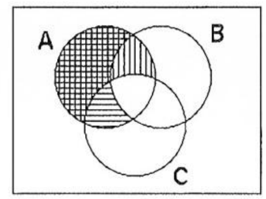
(a)



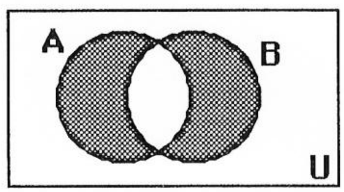
(b)



(c)



34. The figure is as shown; we shade that portion of A that is not in B and that portion of B that is not in A.



35. This is just a restatement of the definition. An element is in $(A \cup B) - (A \cap B)$ if it is in the union (i.e., in either A or B), but not in the intersection (i.e., not in both A and B).