Homework #9, Math 266, Summer 2010

1. In each case, we need to find all the pairs (a, b) with $a \in A$ and $b \in B$ such that the condition is satisfied. This is straightforward.

a) $\{(0,0),(1,1),(2,2),(3,3)\}$ **b)** $\{(1,3),(2,2),(3,1),(4,0)\}$

c) $\{(1,0),(2,0),(2,1),(3,0),(3,1),(3,2),(4,0),(4,1),(4,2),(4,3)\}$

d) Recall that $a \mid b$ means that b is a multiple of a (a is not allowed to be 0). Thus the answer is $\{(1,0),(1,1),(1,2),(1,3),(2,0),(2,2),(3,0),(3,3),(4,0)\}$.

e) We need to look for pairs whose greatest common divisor is 1—in other words, pairs that are relatively prime. Thus the answer is $\{(0,1),(1,0),(1,1),(1,2),(1,3),(2,1),(2,3),(3,1),(3,2),(4,1),(4,3)\}$.

f) There are not very many pairs of numbers (by definition only positive integers are considered) whose least common multiple is 2: only 1 and 2, and 2 and 2. Thus the answer is $\{(1,2),(2,1),(2,2)\}$.

3. a) This relation is not reflexive, since it does not include, for instance $(1,1)$. It is not symmetric, since it includes, for instance, $(2,4)$ but not $(4,2)$. It is not antisymmetric since it includes both $(2,3)$ and $(3,2)$, but $2 \neq 3$. It is transitive. To see this we have to check that *whenever* it includes (a,b) and (b,c), then it also includes (a,c). We can ignore the element 1 since it never appears. If (a,b) is in this relation, then by inspection we see that a must be either 2 or 3. But $(2,c)$ and $(3,c)$ are in the relation for all $c \neq 1$; thus (a,c) has to be in this relation whenever (a,b) and (b,c) are. This proves that the relation is transitive. Note that it is very tedious to prove transitivity for an arbitrary list of ordered pairs.

b) This relation is reflexive, since all the pairs $(1,1)$, $(2,2)$, $(3,3)$, and $(4,4)$ are in it. It is clearly symmetric, the only nontrivial case to note being that both $(1,2)$ and $(2,1)$ are in the relation. It is not antisymmetric because both $(1,2)$ and $(2,1)$ are in the relation. It is transitive; the only nontrivial cases to note are that since both $(1,2)$ and $(2,1)$ are in the relation, we need to have (and do have) both $(1,1)$ and $(2,2)$ included as well.

c) This relation clearly is not reflexive and clearly is symmetric. It is not antisymmetric since both $(2,4)$ and $(4,2)$ are in the relation. It is not transitive, since although $(2,4)$ and $(4,2)$ are in the relation, $(2,2)$ is not.

d) This relation is clearly not reflexive. It is not symmetric, since, for instance, $(1,2)$ is included but $(2,1)$ is not. It is antisymmetric, since there are no cases of (a,b) and (b,a) both being in the relation. It is not transitive, since although $(1,2)$ and $(2,3)$ are in the relation, $(1,3)$ is not.

e) This relation is clearly reflexive and symmetric. It is trivially antisymmetric since there are no pairs (a,b) in the relation with $a \neq b$. It is trivially transitive, since the only time the hypothesis $(a,b) \in R \wedge (b,c) \in R$ is met is when $a = b = c$.

f) This relation is clearly not reflexive. The presence of $(1,4)$ and absence of $(4,1)$ shows that it is not symmetric. The presence of both $(1,3)$ and $(3,1)$ shows that it is not antisymmetric. It is not transitive; both $(2,3)$ and $(3,1)$ are in the relation, but $(2,1)$ is not, for instance.

7. a) This relation is not reflexive since it is not the case that $1 \neq 1$, for instance. It is symmetric: if $x \neq y$, then of course $y \neq x$. It is not antisymmetric, since, for instance, $1 \neq 2$ and also $2 \neq 1$. It is not transitive, since $1 \neq 2$ and $2 \neq 1$, for instance, but it is not the case that $1 \neq 1$.

b) This relation is not reflexive, since $(0,0)$ is not included. It is symmetric, because the commutative property of multiplication tells us that $xy = yx$, so that one of these quantities is greater than or equal to 1 if and only if the other is. It is not antisymmetric, since, for instance, $(2,3)$ and $(3,2)$ are both included. It is transitive. To see this, note that the relation holds between x and y if and only if either x and y are both positive or x and y are both negative. So assume that (a,b) and (b,c) are both in the relation. There are two cases, nearly identical. If a is positive, then so is b, since $(a,b) \in R$; therefore so is c, since $(b,c) \in R$, and hence $(a,c) \in R$. If a is negative, then so is b, since $(a,b) \in R$; therefore so is c, since $(b,c) \in R$, and hence $(a,c) \in R$.

c) This relation is not reflexive, since $(1,1)$ is not included, for instance. It is symmetric; the equation $x = y - 1$ is equivalent to the equation $y = x + 1$, which is the same as the equation $x = y + 1$ with the roles of x and y reversed. (A more formal proof of symmetry would be by cases. If x is related to y then either $x = y + 1$ or $x = y - 1$. In the former case, $y = x - 1$, so y is related to x; in the latter case $y = x + 1$, so y is related to x.) It is not antisymmetric, since, for instance, both $(1,2)$ and $(2,1)$ are in the relation. It is not transitive, since, for instance, although both $(1,2)$ and $(2,1)$ are in the relation, $(1,1)$ is not.

d) Recall that $x \equiv y \pmod 7$ means that $x - y$ is a multiple of 7, i.e., that $x - y = 7t$ for some integer t. This relation is reflexive, since $x - x = 7 \cdot 0$ for all x. It is symmetric, since if $x \equiv y \pmod 7$, then $x - y = 7t$ for some t; therefore $y - x = 7(-t)$, so $y \equiv x \pmod 7$. It is not antisymmetric, since, for instance, we have both $2 \equiv 9$ and $9 \equiv 2 \pmod 7$. It is transitive. Suppose $x \equiv y$ and $y \equiv z \pmod 7$. This means that $x - y = 7s$ and $y - z = 7t$ for some integers s and t. The trick is to add these two equations and note that the y disappears; we get $x - z = 7s + 7t = 7(s+t)$. By definition, this means that $x \equiv z \pmod 7$, as desired.

e) Every number is a multiple of itself (namely 1 times itself), so this relation is reflexive. (There is one bit of controversy here; we assume that 0 is to be considered a multiple of 0, even though we do not consider that 0 is a divisor of 0.) It is clearly not symmetric, since, for instance, 6 is a multiple of 2, but 2 is not a multiple of 6. The relation is not antisymmetric either; we have that 2 is a multiple of -2, for instance, and -2 is a multiple of 2, but $2 \neq -2$. The relation is transitive, however. If x is a multiple of y (say $x = ty$), and y is a multiple of z (say $y = sz$), then we have $x = t(sz) = (ts)z$, so we know that x is a multiple of z.

f) This relation is reflexive, since a and a are either both negative or both nonnegative. It is clearly symmetric from its form. It is not antisymmetric, since 5 is related to 6 and 6 is related to 5, but $5 \neq 6$. Finally, it is transitive, since if a is related to b and b is related to c, then all three of them must be negative, or all three must be nonnegative.

g) This relation is not reflexive, since, for instance, $17 \neq 17^2$. It is not symmetric, since although $289 = 17^2$, it is not the case that $17 = 289^2$. To see whether it is antisymmetric, suppose that we have both (x,y) and (y,x) in the relation. Then $x = y^2$ and $y = x^2$. To solve this system of equations, plug the second into the first, to obtain $x = x^4$, which is equivalent to $x - x^4 = 0$. The left-hand side factors as $x(1-x^3) = x(1-x)(1+x+x^2)$, so the solutions for x are 0 and 1 (and a pair of irrelevant complex numbers). The corresponding solutions for y are therefore also 0 and 1. Thus the only time we have both $x = y^2$ and $y = x^2$ is when $x = y$; this means that the relation is antisymmetric. It is not transitive, since, for example, $16 = 4^2$ and $4 = 2^2$, but $16 \neq 2^2$.

h) This relation is not reflexive, since, for instance, $17 \not\geq 17^2$. It is not symmetric, since although $289 \geq 17^2$, it is not the case that $17 \geq 289^2$. To see whether it is antisymmetric, we assume that both (x,y) and (y,x) are in the relation. Then $x \geq y^2$ and $y \geq x^2$. Since both sides of the second inequality are nonnegative, we can square both sides to get $y^2 \geq x^4$. Combining this with the first inequality, we have $x \geq x^4$, which is equivalent to $x - x^4 \geq 0$. The left-hand side factors as $x(1-x^3) = x(1-x)(1+x+x^2)$. The last factor is always positive, so we can divide the original inequality by it to obtain the equivalent inequality $x(1-x) \geq 0$. Now if $x > 1$ or $x < 0$, then the factors have different signs, so the inequality does not hold. Thus the only solutions are $x = 0$ and $x = 1$. The corresponding solutions for y are therefore also 0 and 1. Thus the only time we have both $x \geq y^2$ and $y \geq x^2$ is when $x = y$; this means that the relation is antisymmetric. It is transitive. Suppose $x \geq y^2$ and $y \geq z^2$. Again the second inequality implies that both sides are nonnegative, so we can square both sides to obtain $y^2 \geq z^4$. Combining these inequalities gives $x \geq z^4$. Now we claim that it is always the case that $z^4 \geq z^2$; if so, then we combine this fact with the last inequality to obtain $x \geq z^2$, so x is related to z. To verify the claim, note that since we are working with integers, it is always the case that $z^2 \geq |z|$ (equality for $z = 0$ and $z = 1$, strict inequality for other z). Squaring both sides gives the desired inequality.

8. We give the simplest example in each case.

a) the empty set on $\{a\}$ (vacuously symmetric and antisymmetric)

b) $\{(a,b),(b,a),(a,c)\}$ on $\{a,b,c\}$

9. The relations in parts **(a)**, **(b)**, and **(e)** all have at least one pair of the form (x, x) in them, so they are not irreflexive. The relations in parts **(c)**, **(d)**, and **(f)** do not, so they are irreflexive.

16. The relations in parts **(a)**, **(b)**, and **(e)** are not asymmetric since they contain pairs of the form (x, x). Clearly the relation in part **(c)** is not asymmetric. The relation in part **(f)** is not asymmetric (both $(1, 3)$ and $(3, 1)$ are in the relation). It is easy to see that the relation in part **(d)** is asymmetric.

20. An asymmetric relation must be antisymmetric, since the hypothesis of the condition for antisymmetry is false if the relation is asymmetric. The relation $\{(a, a)\}$ on $\{a\}$ is antisymmetric but not asymmetric, however, so the answer to the second question is no. In fact, it is easy to see that R is asymmetric if and only if R is antisymmetric and irreflexive.

40. These are just the 16 different subsets of $\{(0,0), (0,1), (1,0), (1,1)\}$.

1. $\emptyset$
2. $\{(0,0)\}$
3. $\{(0,1)\}$
4. $\{(1,0)\}$
5. $\{(1,1)\}$
6. $\{(0,0), (0,1)\}$
7. $\{(0,0), (1,0)\}$
8. $\{(0,0), (1,1)\}$
9. $\{(0,1), (1,0)\}$
10. $\{(0,1), (1,1)\}$
11. $\{(1,0), (1,1)\}$
12. $\{(0,0), (0,1), (1,0)\}$
13. $\{(0,0), (0,1), (1,1)\}$
14. $\{(0,0), (1,0), (1,1)\}$
15. $\{(0,1), (1,0), (1,1)\}$
16. $\{(0,0), (0,1), (1,0), (1,1)\}$

43. This is similar to Example 16 in this section.

a) A relation on a set S with n elements is a subset of $S \times S$. Since $S \times S$ has n^2 elements, we are asking for the number of subsets of a set with n^2 elements, which is 2^{n^2}. In our case $n = 4$, so the answer is $2^{16} = 65{,}536$.

b) In solving part **(a)**, we had 16 binary choices to make—whether to include a pair (x, y) in the relation or not as x and y ranged over the set $\{a, b, c, d\}$. In this part, one of those choices has been made for us: we *must* include (a, a). We are free to make the other 15 choices. So the answer is $2^{15} = 32{,}768$. See Exercise 45 for more problems of this type.

1. In each case we need to check for reflexivity, symmetry, and transitivity.

a) This is an equivalence relation; it is easily seen to have all three properties. The equivalence classes all have just one element.

b) This relation is not reflexive since the pair $(1, 1)$ is missing. It is also not transitive, since the pairs $(0, 2)$ and $(2, 3)$ are there, but not $(0, 3)$.

c) This is an equivalence relation. The elements 1 and 2 are in the same equivalence class; 0 and 3 are each in their own equivalence class.

d) This relation is reflexive and symmetric, but it is not transitive. The pairs $(1, 3)$ and $(3, 2)$ are present, but not $(1, 2)$.

e) This relation would be an equivalence relation were the pair $(2, 1)$ present. As it is, its absence makes the relation neither symmetric nor transitive.

2. **a)** This is an equivalence relation by Exercise 9 ($f(x)$ is x's age).

b) This is an equivalence relation by Exercise 9 ($f(x)$ is x's parents).

c) This is not an equivalence relation, since it need not be transitive. (We assume that biological parentage is at issue here, so it is possible for A to be the child of W and X, B to be the child of X and Y, and C to be the child of Y and Z. Then A is related to B, and B is related to C, but A is not related to C.)

d) This is not an equivalence relation since it is clearly not transitive.

e) Again, just as in part **(c)**, this is not transitive.

10. The function that sends each $x \in A$ to its equivalence class $[x]$ is obviously such a function.

15. By algebra, the given condition is the same as the condition that $f((a,b)) = f((c,d))$, where $f((x,y)) = x-y$. Therefore by Exercise 9 this is an equivalence relation. If we want a more explicit proof, we can argue as follows. For reflexivity, $((a,b),(a,b)) \in R$ because $a+b = b+a$. For symmetry, $((a,b),(c,d)) \in R$ if and only if $a+d = b+c$, which is equivalent to $c+b = d+a$, which is true if and only if $((c,d),(a,b)) \in R$. For transitivity, suppose $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$. Thus we have $a+d = b+c$ and $c+e = d+f$. Adding, we obtain $a+d+c+e = b+c+d+f$. Simplifying, we have $a+e = b+f$, which tells us that $((a,b),(e,f)) \in R$.

36. Only part **(a)** and part **(c)** are equivalence relations. In part **(a)** each element is in an equivalence class by itself. In part **(c)** the elements 1 and 2 are in one equivalence class, and 0 and 3 are each in their own equivalence class.

41. The sets in a partition must be nonempty, pairwise disjoint, and have as their union all of the underlying set.

a) This is not a partition, since the sets are not pairwise disjoint (the elements 2 and 4 each appear in two of the sets).

b) This is a partition. **c)** This is a partition.

d) This is not a partition, since none of the sets includes the element 3.

46. **a)** This is a partition, since it satisfies the definition.

b) This is a partition, since it satisfies the definition.

c) This is not a partition, since the intervals are not disjoint (they share endpoints).

d) This is not a partition, since the union of the subsets leaves out the integers.

e) This is a partition, since it satisfies the definition.

f) This is a partition, since it satisfies the definition. Each equivalence class consists of all real numbers with a fixed fractional part.