Chain Rule

The chain rule for multiple variables is used when a function of two or more variables are in-turn defined in terms of one or more variables. For instance if w=f(x,y) is our function, and x=x(t), and y=y(t), then the chain rules looks like:

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}$$

If w=f(x,y,z), and x=x(t), and y=y(t), z=z(t), then this becomes:

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

We can extend this to cases where w=f(x,y) and x=x(t,s), and y=y(t,s) for the partial derivatives of w.

$$\frac{\partial w}{\partial t}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$
$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

Likewise, we can extend this to w=f(x,y,z), and x=x(t,s), y=y(t,s) and z=z(t,s) and so forth.

$$\frac{\partial w}{\partial t}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$
$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

In practice, it's usually best to take all the derivative components separately and then put them together. (You won't be asked to simplify the most complicated problems.) While it is possible to replace x and y (and z) into the w function and take the derivative by the usual chain rule, this can be quite complicated because what might just follow the formula nicely as above, could turn into multiple chain and product rules smashed together if you do the substitution first. While the expressions will be algebraically equivalent, substituting first can make things harder to derive.

Example 1. Find $\frac{dw}{dt}$ for $w = xy + xz + yz, x = t - 1, y = t^2 - 1, z = t$.
Start by finding out all the derivatives and partial derivatives you'll need:

$$\frac{\partial w}{\partial x} = y + z = t^2 - 1 + t = t^2 + t - 1 \qquad \frac{dx}{dt} = 1$$

$$\frac{\partial w}{\partial y} = x + z = t - 1 + t = 2t - 1 \qquad \frac{dy}{dt} = 2t$$

$$\frac{\partial w}{\partial z} = x + y = t - 1 + t^2 - 1 = t^2 + t - 2 \qquad \frac{dx}{dt} = 1$$

Once everything is replaced in terms of the t variable, you can apply the formula:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = \left(t^2 + t - 1\right)1 + \left(2t - 1\right)2t + \left(t^2 + t - 2\right)1 =$$
$$t^2 + t - 1 + 4t^2 - 2t + t^2 + t - 2 = 6t^2 - 3$$

For some simpler cases like this one, it is possible to plug in the original functions and check.

$$w = xy + xz + yz = (t-1)(t^2-1) + (t-1)t + (t^2-1)t =$$
$$t^3 - t^2 - t + 1 + t^2 - t + t^3 - t = 2t^3 - 3t + 1$$

From this we can verify that $\frac{dw}{dt} = 6t^2 - 3$.

This was a simply polynomial. When we start mixed multiple parameters and function types, it is generally best to use the chain rule and not attempt to do substitution first.

One important point to note: *it is not appropriate to get an answer for this problem that contains three variables: x, y, and t. By the end of the problem, all the x and y variables* ***MUST*** *be replaced by their expressions in terms of t, whether you are simplifying them further or not.*

Example 2. Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$ for $w = x^2 + y^2 + z^2, x = tsin(s), y = tcos(s), z = st^2$.
As before, it's easiest to take all your derivatives first, and make your substitutions into the derivatives.

$$\frac{\partial w}{\partial x} = 2x = 2tsin(s) \qquad \frac{\partial x}{\partial t} = \sin(s) \qquad \frac{\partial x}{\partial s} = t\cos(s)$$
$$\frac{\partial w}{\partial y} = 2y = 2tcos(s) \qquad \frac{\partial y}{\partial t} = \cos(s) \qquad \frac{\partial y}{\partial s} = -t\sin(s)$$
$$\frac{\partial w}{\partial z} = 2z = 2st^2 \qquad \frac{\partial z}{\partial t} = 2\text{st} \qquad \frac{\partial z}{\partial s} = t^2$$

According to our chain rules:

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} = 2t\sin(s)\sin(s) + 2t\cos(s)\cos(s) + 2st^2 2st =$$
$$2t\sin^2 s + 2t\cos^2 s + 4s^2t^3 = 2t + 4s^2t^3$$

And

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = 2t\sin(s)t\cos(s) + 2t\cos(s)\left(-t\sin(s)\right) + 2st^2t^2 = 2st^4$$

Practice Problems.

1. Find $\frac{dw}{dt}$ for the following sets of equations using the chain rule. Be sure your final answers contain only t.
   a. $w = \sqrt{x^2 + y^2}, x = \cos(t), y = e^t$
   b. $w = xsin(y), x = e^t, y = \pi - t$
   c. $w = \cos(x - y), x = t^2, y = 1$
   d. $w = xy^2 + x^2z + yz^2, x = t^2, y = \arccos(t),\ z = e^{-2t}$
2. Find $\frac{d^2w}{dt^2}$ for problems 1b, and 1c.
3. Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$ for the following sets of equations using the chain rule. Be sure your final answer contains only t and s.
   a. $w = x^2 + y^2, x = s + t, y = s - t$
   b. $w = y^3 - 3x^2y, x = e^s, y = e^{t^2}$
   c. $w = xyz, x = s + t, y = s - t, z = st^2$
   d. $w = ze^{xy}, x = s - t, y = s + t, z = st$
4. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ for each of the following sets of equations using the chain rule. Be sure your final answer contains only r and θ.
   a. $w = x^2 - 2xy + y^2, x = r + \theta, y = r - \theta$
   b. $w = \sqrt{25 - 5x^2 - 5y^2}, x = rcos(\theta), y = rsin(\theta)$