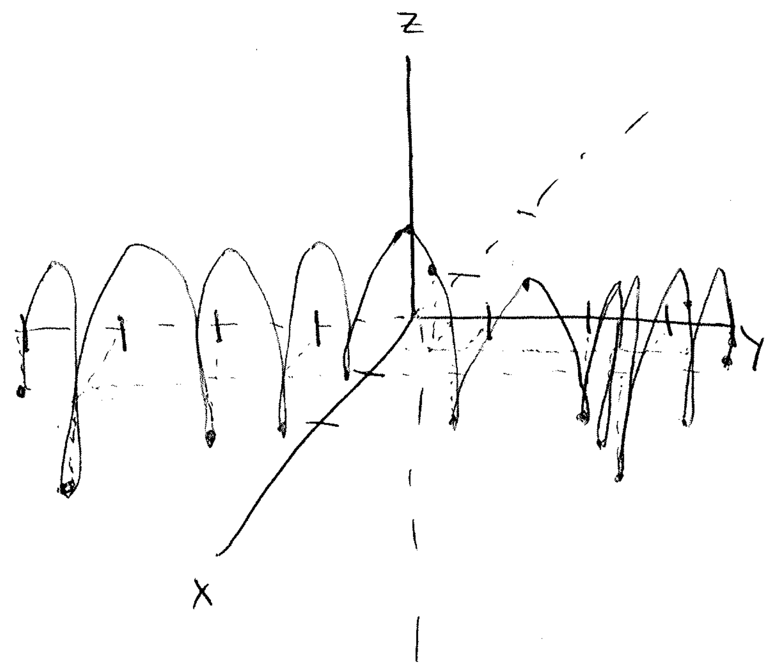
Name KEY

Math 254, Exam #1, Summer 2012

Instructions: Show all work. Answers with no work will be graded all or nothing unless the point of the problem is to show the work (in which case, no work will receive no credit). Use exact values (fractions and square roots, etc.) unless the problem tells you to round, is a word problem, or begins with decimal values.

1. Graph the following parametric equation in 3D: $\overrightarrow{r(t)} = \sin^2(t)\hat{i} + t\hat{j} + \cos(t^2)\hat{k}$. Try t={-2π, -3π/2, -π, -π/2, 0, π/2, π, 3π/2, 2π} or if the graph is less predictable, try t={0, π/4, π/2, 3π/4, π, 5π/4, 3π/2, 7π/4, 2π}. Be sure to label each graph with an arrow on the curve in which t is increasing. You need to show enough to know what the equation of the graph looks like. For a helix, this means two spirals. Label all three axes. (20 points)

t	x	y	z
-2π	0	-2π	$-.21$
$-3\pi/2$	1	$-3\pi/2$	$-.97$
$-\pi$	0	$-\pi$	$-.9$
$-\pi/2$	1	$-\pi/2$	$-.78$
0	0	0	1
$\pi/2$	1	$\pi/2$	$-.78$
π	0	π	$-.9$
$3\pi/2$	1	$3\pi/2$	$-.97$
2π	0	2π	$-.21$
$\pi/4$	$1/2$	$\pi/4$	$.82$
$3\pi/4$	$1/2$	$3\pi/4$	$.74$
$5\pi/4$	$1/2$	$5\pi/4$	$-.96$
$7\pi/4$	$1/2$	$7\pi/4$	$.37$



2. For the vector-valued function $\vec{r}(t) = e^t\vec{i} + \sec^2 t\,\vec{j} + \frac{1}{t^2+1}\vec{k}$, find $\vec{r}'(t)$ and $\int \vec{r}(t)dt$. (20 points)

$$r'(t) = e^t\hat{i} + 2\sec^2 t \tan t\,\hat{j} + \frac{-2t}{(t^2+1)^2}\hat{k}$$

$$\int r(t)\,dt = (e^t + C_1)\hat{i} + (\tan t + C_2)\hat{j} + (\arctan t + C_3)\hat{k}$$

3. Given the acceleration function $\vec{a}(t) = -\cos t\vec{j} - \sin t\vec{k}$, and initial conditions $\vec{v}(0) = 2\vec{i} + \vec{k}, \vec{r}(0) = 3\vec{j}$, find the velocity and position functions. (15 points)

$v(t) = \int -\cos t\,\hat{j} - \sin t\,\hat{k}\,dt = C_1\hat{i} - (\sin t + C_2)\hat{j} + (\cos t + C_3)\hat{k}$

$C_1 = 2 \quad C_2 = 0 \quad C_3 = 0$

$r(t) = \int 2\hat{i} - \sin t\,\hat{j} + \cos t\,\hat{k}\,dt = (2t + C_1)\hat{i} + (\cos t + C_2)\hat{j} + (\sin t + C_3)\hat{k}$

$C_1 = 0 \quad C_2 = 2 \quad C_3 = 0$

$$\boxed{r(t) = 2t\hat{i} + (\cos t + 2)\hat{j} + (\sin t)\hat{k}}$$

4. Find the unit tangent and unit normal vectors for the curve $\vec{r}(t) = 2\sin t\vec{i} + 2\cos t\vec{j} + 4\vec{k}$ at the point $P(\sqrt{2}, \sqrt{2}, 4)$. (20 points)

$t = \pi/4$

$r'(t) = 2\cos t\,\hat{i} - 2\sin t\,\hat{j}$

$\|r'(t)\| = 2$

$T(t) = \cos t\,\hat{i} - \sin t\,\hat{j}$

$T'(t)$

$N(t) = -\sin t\,\hat{i} - \cos t\,\hat{j}$ ← magnitude already 1

$T(\pi/4) = \frac{\sqrt{2}}{2}\hat{i} - \frac{\sqrt{2}}{2}\hat{j}$

$N(\pi/4) = -\frac{\sqrt{2}}{2}\hat{i} - \frac{\sqrt{2}}{2}\hat{j}$

5. Find the curvature K of the curve. $\bar{r}(t) = 4(\sin t - t\cos t)\bar{i} + 4(\cos t + t\sin t)\bar{j} + \frac{2}{3}t^2\bar{k}$ (10 points)

$$r'(t) = 4(\cancel{\cos t} - \cancel{\cos t} + t\sin t)\hat{i} + 4(\cancel{-\sin t} + \cancel{\sin t} + t\cos t)\hat{j} + \frac{4}{3}t\hat{k}$$

$$4t\sin t\,\hat{i} \quad + \quad 4t\cos t\,\hat{j} \quad + \quad \frac{4}{3}t\,\hat{k}$$

$$r''(t) = (4\sin t + 4t\cos t)\hat{i} + (+4\cos t - 4t\sin t)\hat{j} + \frac{4}{3}k$$

$$r' \times r'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4t\sin t & 4t\cos t & \frac{4}{3}t \\ 4(\sin t + t\cos t) & 4(\cos t - t\sin t) & \frac{4}{3} \end{vmatrix}$$

$$\left(\cancel{\tfrac{16}{3}t\cos t} - \cancel{\tfrac{16}{3}t\cos t} + \tfrac{16}{3}t^2\sin t\right)\hat{i} -$$

$$\left(\cancel{\tfrac{16}{3}t\sin t} - \cancel{\tfrac{16}{3}t\sin t} - \tfrac{16}{3}t^2\cos t\right)\hat{j} +$$

$$\left(\cancel{16t\sin t\cos t} - 16t^2\sin^2 t - \cancel{16t\sin t\cos t} - 16t^2\cos^2 t\right)\hat{k}$$

$$= \frac{16}{3}t^2\sin t\,\hat{i} + \frac{16}{3}t^2\cos t\,\hat{j} - 16t^2\,\hat{k}$$

$$\|r' \times r''\| = \sqrt{\tfrac{256}{9}t^4 + 256t^4}$$

$$\|r'(t)\| = \sqrt{16t^2 + \tfrac{16}{9}t^2}$$

$$\boxed{K = \frac{\sqrt{\left(\frac{256}{9} + 256\right)t^4}}{\left(\sqrt{\left(\frac{16}{9} + 16\right)t^2}\right)^3}} = \frac{\cancel{t}\,\frac{16}{3}\,10^{1/2}}{4\,\frac{4^3}{2^3}\,10^{3/2}} =$$

$$\frac{9}{4t \cdot 10} = \frac{9}{40t}$$

6. Find the limits. (10 points each)

a. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$

$$\lim_{r\to 0} \frac{\cancel{r^2}\sin\theta\cos\theta}{\cancel{r^2}} = \sin\theta\cos\theta = \text{DNE}$$

b. $\lim_{(x,y)\to(0,0)} \frac{2x^3y^2}{x^6 + y^4}$

$y^2 = x^3$

$$\lim_{x\to 0} \frac{2x^3x^3}{x^6 + x^6} = \frac{2x^6}{2x^6} = 1$$

$y=0$

$$\lim_{x\to 0} \frac{2x^3(0)}{x^6 + 0} = \frac{0}{x^6} = 0$$

DNE

7. Find the partial derivatives w_x, w_y, w_z of the function $w = 3x^2y - 5xyz + 10yz^2$ at the point (-1,1,4). (15 points)

$w_x = 6xy - 5yz$

$w_y = 3x^2 - 5xz + 10z^2$

$w_z = -5xy + 20yz$

$w_x(-1,1,4) = 6(-1)(1) - 5(1)(4) = -6 - 20 = -26$

$w_y(-1,1,4) = 3(-1)^2 - 5(-1)(4) + 10(4)^2 = 3 + 20 + 160 = 183$

$w_z(-1,1,4) = -5(-1)(1) + 20(1)(4) = 5 + 80 = 85$

8. Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$ for the following set of equations using the chain rule:
$w = y^3 - 3x^2y, x = e^s, y = e^{t^2}$
Be sure your final answer contains only t and s. (20 points)

$\frac{\partial w}{\partial x} = 6xy$ $\quad \frac{\partial x}{\partial t} = 0$ $\quad \frac{\partial y}{\partial t} = 2te^{t^2}$

$\frac{\partial w}{\partial y} = 3y^2 - 3x^2$ $\quad \frac{\partial x}{\partial s} = e^s$ $\quad \frac{\partial y}{\partial s} = 0$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = 6(e^s)(e^{t^2})(0) + (3e^{2t^2} - 3e^{2s})(2te^{t^2})$$
$$= \boxed{3(e^{2t^2} - e^{2s})2te^{t^2}}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = 6(e^s)(e^{t^2})(e^s) + (3e^{2t^2} - 3e^{2s})(0)$$
$$= \boxed{6e^s e^{t^2} e^s}$$

9. Find the implicit derivative, or first partial derivatives, z_x, z_y of the following implicit function $x^5 - xyz + z^4 = \ln\left(\frac{x}{y}\right)$. You must find at least one of the derivatives implicitly, but you may use the formulas for the second one if you wish. (20 points)

$\ln x - \ln y$

z_x: $5x^4 - yz - xyz_x + 4z^3 z_x = \frac{1}{x}$

$z_x(-xy+4z^3) = \frac{1}{x} + yz - 5x^4$

$$\boxed{z_x = \frac{\frac{1}{x} + yz - 5x^4}{-xy+4z^3}}$$

$z_y: = \frac{-F_y}{F_z}$ $\quad F(x,y,z) = x^5 - xyz + z^4 - \ln x + \ln y = 0$

$F_y = -xz + \frac{1}{y}$

$F_z = -xy + 4z^3$

$$\boxed{z_y = -\frac{xz + \frac{1}{y}}{-xy+4z^3} = \frac{-xz - \frac{1}{y}}{-xy+4z^3}}$$

10. Find f_x, f_{xx}, f_{xy}, f_y, f_{yy} for the function $z = e^y \sin xy$. Be wary: product rules are involved. (20 points)

$f_x = e^y \cos xy \cdot y$

$f_y = e^y \sin xy + e^y \cos xy \cdot x = e^y(\sin xy + x\cos xy)$

$f_{xx} = -e^y y \sin xy \cdot y = -e^y y^2 \sin xy$

$f_{yy} = e^y(\sin xy + x\cos xy) + e^y(x\cos xy + -x^2 \sin xy)$

$f_{xy} = e^y \cos xy \cdot y + -e^y \sin xy \cdot xy + e^y \cos xy$

11. Find the gradient, ∇f, for each function with the appropriate gradient formula. (10 points each)

a. $f(x,y,z) = xy^2 + x^2z + yz^2$

$$\nabla f = \langle y^2 + 2xz,\ 2xy + z^2,\ x^2 + 2yz \rangle$$

b. $f(r,\theta,z) = r^3z - \dfrac{6}{1 - r\cos\theta}$ using $\nabla = \left\langle \dfrac{\partial}{\partial r}, \dfrac{1}{r}\cdot\dfrac{\partial}{\partial \theta}, \dfrac{\partial}{\partial z} \right\rangle$

$-6(1-r\cos\theta)^{-1}$

$$\nabla f = \langle 3r^2z + 6(-1)(1-r\cos\theta)^{-2}(-\cos\theta),\ \tfrac{1}{\cancel{r}}\cdot(-6)(-1)(1-r\cos\theta)^{-2}(+\cancel{r}\sin\theta),\ r^3 \rangle$$

$$= \left\langle 3r^2z - \frac{6\cos\theta}{(1-r\cos\theta)^2},\ \frac{6\sin\theta}{(1-r\cos\theta)^2},\ r^3 \right\rangle$$

12. Find the directional derivative of the given function $f(x,y,z) = xy + yz + xz$ at the point P(1,1,1) in the direction $\vec{v} = 2\vec{i} + \vec{j} - \vec{k}$. (20 points)

$$\vec{u} = \frac{2}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k} \qquad \sqrt{2^2+1^2+1^2} = \sqrt{6}$$

$$\nabla f = \langle y+z,\ x+z,\ y+\cancel{z} \rangle \Rightarrow \nabla f(1,1,1) = \langle 2,2,2 \rangle$$

$$\langle 2,2,2 \rangle \cdot \left\langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle =$$

$$\frac{4}{\sqrt{6}} + \cancel{\frac{2}{\sqrt{6}}} - \cancel{\frac{2}{\sqrt{6}}} = \boxed{\frac{4}{\sqrt{6}}}$$