Name KEY

Math 254, Exam #2, Summer 2012

Instructions: Show all work. Answers with no work will be graded all or nothing unless the point of the problem is to show the work (in which case, no work will receive no credit). Use exact values (fractions and square roots, etc.) unless the problem tells you to round, is a word problem, or begins with decimal values.

1. Find the equations of the tangent plane and the normal line to the curve $2xy - z^3 = 0, P(2,2,2)$ at the given point. (20 points)

$\nabla F = \langle 2y, 2x, -3z^2 \rangle$

$= \langle 4, 4, -12 \rangle$

plane: $4(x-2) + 4(y-2) - 12(z-2) = 0$

line: $\frac{x-2}{4} = \frac{y-2}{4} = \frac{z-2}{-12}$

2. Find all the critical points of the function $f(x,y) = 2x^2 + 2xy + y^2 + 2x - 3$, and use the second partials test to determine whether each critical point is a maximum, a minimum, a saddle point, or if the test fails. (20 points)

$f_x = 4x + 2y + 2 = 0 \Rightarrow 4(-y) + 2y = -2$

$f_y = 2x + 2y = 0 \Rightarrow x = -y \quad -2y = -2$

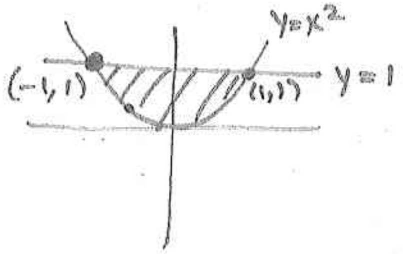
$\boxed{\begin{array}{l} y = 1 \\ x = -1 \end{array}}$

$f_{xx} = 4$

$f_{yy} = 2 \quad 4(2) - 2^2 = 8 - 4 = 4 > 0$

$f_{xy} = 2 \quad f_{xx} = 4 > 0 \quad \cup$ minimum

3. Find the absolute extrema for each function on the indicated region:
$f(x,y)=2x-2xy+y^2, R:\{(x,y) \mid y \geq x^2, y \leq 1\}$ (30 points)



$f_x = 2-2y=0$

$2y=2 \quad y=1$

$f_y = 2x+2y=0$

$x=-y$

$x=-1$

$(-1,1)$

test pts.

$(-1,1)$

$(1,1)$

$(-1/2, 1/4)$

$y=x^2$

$f(x)= 2x-2x(x^2)+(x^2)^2$

$2x-2x^3+x^4$

$2-6x^2+4x^3=0$

$x=-1/2$

$x=1$

$y=1$ $\quad$ $y=\frac{1}{4}$

$f(-1,1)=$

$2(-1)-2(-1)(1)+(1)^2=$

$-2+2+1=1$

$f(1,1) = 2(1)-2(1)(1)+1^2$

$2-2+1=1$

$y=1$

$f(x)=2x-2x+1=1$

$f'(0)=0$

$f(-\frac{1}{2},\frac{1}{4}) = 2(-\frac{1}{2})-2(-\frac{1}{2})(\frac{1}{4}) + (\frac{1}{4})^2$

$=-11/16$

Corner points already used

Max $f(-1,1)=f(1,1)=1$

Min $f(-\frac{1}{2},\frac{1}{4}) = -11/16$

4. Maximize $w=x^2-10x+y^2-14y+28$, subject to $x+y=10$ (20 points)

$F = x^2-10x+y^2-14y+28-\lambda x-\lambda y+10\lambda$

$F_x = 2x-10-\lambda=0$ $\qquad \lambda = 2x-10$

$F_y = 2y-14-\lambda=0$ $\qquad \lambda = 2y-14$

$F_\lambda: x+y=10$

$\Downarrow$

$2x-10=2y-14$

$x-5=y-7$

$+7 \quad +7$

$x+2=y$

$x+(x+2)=10$

$2x+2=10$

$2x=8$

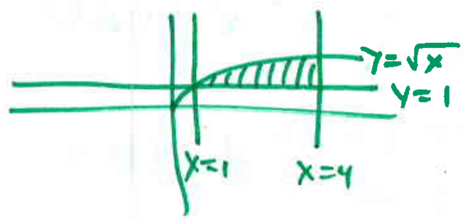
$x=4$

$y=6$

$\boxed{(4,6) \quad w=-44}$

5. Evaluate the integral. Sketch or describe the region. (15 points each)

a. $\int_1^4\int_1^{\sqrt{x}} 2ye^{-x}\,dy\,dx$



$y^2e^{-x}\Big|_1^{\sqrt{x}}$

$xe^{-x}-e^{-x}$

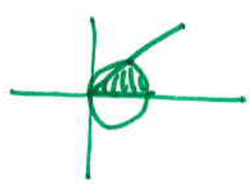
$\int_1^4 (x-1)e^{-x}\,dx$ $\qquad u=x-1 \quad e^{-x}=dv$

$du=dx \quad -e^{-x}=v$

$-(x-1)e^{-x}+\int e^{-x}dx = -(x-1)e^{-x}-e^{-x}\Big|_1^4 = -3e^{-4}-e^{-4}+0+e^{-1} = \boxed{-4e^{-4}+e^{-1}}$

b. $\int_0^{\pi/4}\int_0^{\cos\theta} 3r^2\sin\theta\,dr\,d\theta$

$r^3\Big|_0^{\cos\theta} \sin\theta$



$\int_0^{\pi/4} \cos^3\theta\sin\theta\,d\theta$ $\qquad u=\cos\theta$

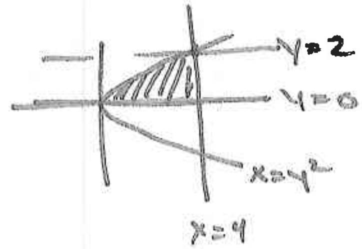
$-du=\sin\theta$

$-\frac{\cos^4\theta}{4}\Big|_0^{\pi/4} = -\frac{1}{4}\left[\frac{1}{4}-1\right] = -\frac{1}{4}\left(-\frac{3}{4}\right) = \boxed{+\frac{3}{16}}$

c. $\displaystyle\int_0^2\int_{y^2}^4 \sqrt{x}\sin x\,dx\,dy$ (It will be necessary to switch the order of integration here.)

$x=4$

$x=y^2$



$y=\sqrt{x}$
$y=4$
$y=0$
$x=0$

$$\int_0^4\int_0^{\sqrt{x}} \sqrt{x}\sin x\,dy\,dx = \int_0^4 y\sqrt{x}\sin x\Big|_0^{\sqrt{x}}dx$$

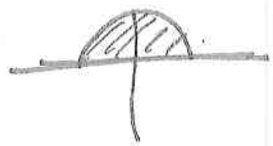
$$\int_0^4 x\sin x\,dx \qquad u=x \quad dv=\sin x\,dx \qquad du=dx \quad v=-\cos x$$

$$-x\cos x+\int\cos x\,dx = -x\cos x+\sin x\Big|_0^4 = -4\cos 4+\sin 4-(0+0)$$

$$\boxed{-4\cos(4)+\sin 4}$$

6. Set up an integral in polar coordinates and evaluate the integral over the region R. Sketch the region. (15 points)

$\iint_R (x^2+y^2)\,dA$; R: semicircle bounded by $y=\sqrt{4-x^2}, y=0$



$r=2$

$\theta=\pi$

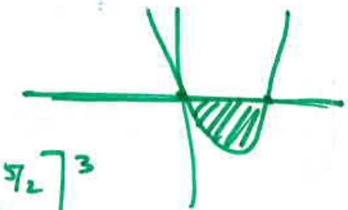
$$\int_0^{\pi}\int_0^2 r^2\,r\,dr\,d\theta$$

$$\int_0^{\pi}\int_0^2 r^3\,dr\,d\theta = \int_0^{\pi}\frac{r^4}{4}\Big|_0^2 d\theta = \int_0^{\pi}\frac{16}{4}\,d\theta = \int_0^{\pi}4\,d\theta$$

$$=\boxed{4\pi}$$

7. Find the mass and center of mass of the lamina bounded by the graphs of the equations for the given density: $y = x^2 - 3x, y = 0, \rho = k\sqrt{x}$. (30 points)

$$M = \int_0^3 \int_{x^2-3x}^{0} k\sqrt{x}\, dy\, dx$$



$$k\int_0^3 -x^{1/2}(x^2-3x)\,dx$$

$$-k\int_0^3 x^{5/2} - 3x^{3/2}\,dx = -k\left[\frac{2}{7}x^{7/2} - 3\cdot\frac{2}{5}x^{5/2}\right]_0^3$$

$$-k\left[\frac{2}{7}3^{7/2} - \frac{6}{5}3^{5/2}\right] = 3^{1/2}k\left[\frac{2}{5}3^3 - \frac{2}{7}3^3\right] = \frac{108\sqrt{3}k}{35}$$

$$M_x = \int_0^3\int_{x^2-3x}^{0} k\sqrt{x}\, y\, dy\, dx = -k\int_0^3 \frac{x^{1/2}}{2}(x^2-3x)^2\,dx =$$

$$\int_0^3 -\frac{k}{2}(x^{1/2})(x^4 - 6x^3 + 9x^2)\,dx = -\frac{k}{2}\int_0^3 x^{9/2} - 6x^{7/2} + 9x^{5/2}\,dx$$

$$-\frac{k}{\cancel{2}}\left[\frac{\cancel{2}}{11}x^{11/2} - 6\cdot\frac{\cancel{2}}{9}x^{9/2} + 9\cdot\frac{2}{7}x^{7/2}\right]_0^3 = -\frac{k\sqrt{3}}{\cancel{2}}\left[\frac{3^5}{11} - \frac{6}{9}3^4 + \frac{9}{7}3^3\right] = \frac{-216\sqrt{3}k}{77}$$

$$M_y = \int_0^3\int_{x^2-3x}^{0} kx^{3/2}\,dy\,dx = k\int_0^3 x^{3/2}(x^2-3x)\,dx = -k\int_0^3 x^{7/2} - 3x^{5/2}\,dx$$

$$-k\left[\frac{2}{9}3^{9/2} - 3\cdot\frac{2}{7}3^{7/2}\right] = -k\sqrt{3}\left[\frac{2}{9}\cdot 3^4 - 3\cdot\frac{2}{7}3^3\right] = \frac{36\sqrt{3}k}{7}$$

$$\bar{x} = \frac{M_y}{M} = \frac{36\sqrt{3}k}{7}\cdot\frac{35}{108\sqrt{3}k} = \frac{5}{3}$$

$$\bar{y} = \frac{M_x}{M} = \frac{-216\sqrt{3}k}{77}\cdot\frac{35}{108\sqrt{3}k} = -\frac{10}{11}$$

$$(\bar{x}, \bar{y}) = \left(\frac{5}{3}, -\frac{10}{11}\right)$$

8. Find the area of the surface given by f (or z) over the region R:
$f(x,y)=e^{-x}\sin y; R=\{(x,y): 0\le x\le 4, 0\le y\le x\}$. (30 points)

$$S = \iint_R \sqrt{1+(f_x)^2+(f_y)^2}\, dA \qquad f_x = -e^{-x}\sin y \qquad f_y = e^{-x}\cos y$$

$$S = \iint_R \sqrt{1+e^{-2x}\sin^2 y + e^{-2x}\cos^2 y}\, dA =$$

$$S = \iint_R \sqrt{1+e^{-2x}}\, dA = \int_0^4 \int_0^x \sqrt{1+e^{-2x}}\, dy\, dx =$$

$$\int_0^4 y\sqrt{1+e^{-2x}}\Big|_0^x dx = \int_0^4 x\sqrt{1+e^{-2x}}\, dx \approx 8.11811$$

9. The sum of three numbers is 60. Maximize the product of the three numbers. (10 points)

$x+y+z=60$

$F(x,y,z) = xyz$

$w = xyz - \lambda x - \lambda y - \lambda z + 60\lambda$

$w_x = yz - \lambda = 0 \qquad \lambda = yz$

$w_y = xz - \lambda = 0 \qquad \lambda = xz$

$w_z = xy - \lambda = 0 \qquad \lambda = xy$

$yz - xz = 0 \qquad xz = xy$

$z(y-x) = 0 \qquad xz - xy = 0$

$z=0 \quad y=x \qquad x(z-y)=0$

$x=0 \quad z=y$

zero's won't produce max product, so ignore

$y=x, \; z=y \Rightarrow$

$x=y=z$

$x+\cancel{x}+x=66$

$3x=60$

$\boxed{x=20 \quad y=20 \quad z=20}$