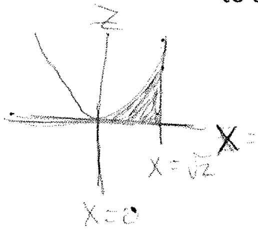
Name KEY

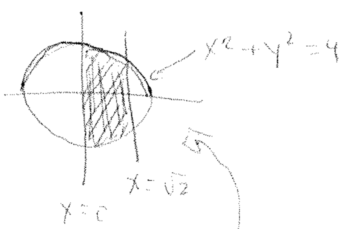
Math 254, Final Exam, Summer 2012

Instructions: Show all work. Answers with no work will be graded all or nothing unless the point of the problem is to show the work (in which case, no work will receive no credit). Use exact values (fractions and square roots, etc.) unless the problem tells you to round, is a word problem, or begins with decimal values.

1. Evaluate the integral $\int_0^{\sqrt{2}}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{x^2} y\,dz\,dy\,dx$. Sketch or describe the region. It may be convenient to switch to cylindrical coordinates. (15 points)



$$\int_0^{\sqrt{2}}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x^2 y\,dy\,dx$$



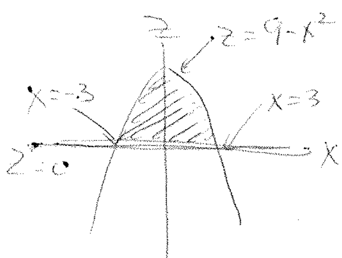
$$\int_0^{\sqrt{2}} \frac{1}{2}y^2x^2\Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = \int_0^{\sqrt{2}} \frac{1}{2}x^2\underbrace{\left(\left(\sqrt{4-x^2}\right)^2-\left(-\sqrt{4-x^2}\right)^2\right)}_{=0}dx$$

$$= 0$$

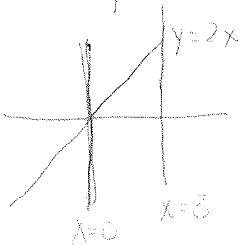
because of this, you'll need 2 integrals in cylindrical

may be best to stay in rectangular

2. Set up an integral to find the volume of the solid bounded by $z = 9-x^2, z = 0, x = 0$ and $y = 2x$. (15 points)



$$\int_0^3\int_0^{2x}\int_0^{9-x^2} dz\,dy\,dx$$



3. For the function f(x,y) calculate the region below it on the region described. Do this by changing to a convenient pair of variables. Sketch the region before and after the switch.
$f(x,y) = e^{-\frac{xy}{2}}$ on the region bounded by $y = 2x, y = \frac{4}{x}, y = \frac{1}{4}x, y = \frac{1}{x}$. (30 points)

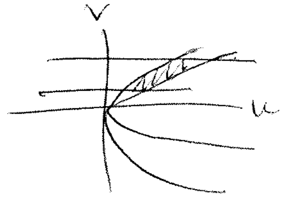
$x = \frac{u}{v} \leftarrow xy = u$

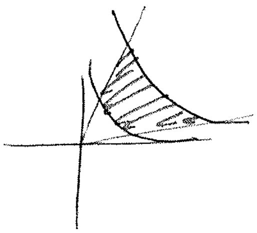
$y = v$

$v = 2\frac{u}{v}$ $\quad v = \frac{1}{4}\frac{u}{v}$

$v^2 = 2u$ $\quad v^2 = \frac{1}{4}u$

$u = \frac{v^2}{2}$ $\quad u = 4v^2$





$1 \le u \le 4$

$\frac{\sqrt{u}}{2} \le v \le \sqrt{2u}$

$$\int_1^4 \int_{\frac{\sqrt{u}}{2}}^{\sqrt{2u}} e^{-\frac{u}{2}} \cdot \frac{1}{v} \, dv\, du \qquad \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v} + 0$$

$$\int_1^4 \ln|v| \Big|_{\frac{\sqrt{u}}{2}}^{\sqrt{2u}} e^{-\frac{u}{2}} \, du$$

$$= \ln|\sqrt{2u}| - \ln\left|\frac{\sqrt{u}}{2}\right| = \ln\left|\frac{\sqrt{2u}}{\frac{\sqrt{u}}{2}}\right| = \ln\left|\frac{\sqrt{2u}\cdot 2}{\sqrt{u}}\right| = \ln 2\sqrt{2}$$

$$\ln(2\sqrt{2})\int_1^4 e^{-\frac{u}{2}}\, du = \ln(2\sqrt{2})(-2)e^{-u/2}\Big|_1^4 =$$

$$-2\ln(2\sqrt{2})\left[e^{-2} - e^{-1/2}\right] = \boxed{\ln\left(\tfrac{1}{8}\right)\left[\tfrac{1}{e^2} - \tfrac{1}{\sqrt{e}}\right]}$$

Or use $xy = u$

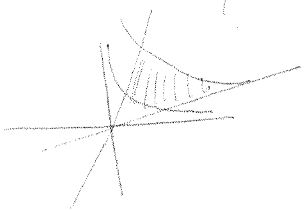
$\frac{y}{x} = v$

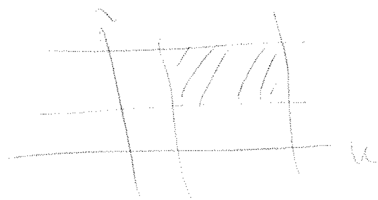
3. For the function f(x,y) calculate the region below it on the region described. Do this by changing to a convenient pair of variables. Sketch the region before and after the switch.
$f(x,y) = e^{-\frac{xy}{2}}$ on the region bounded by $y = 2x, y = \frac{4}{x}, y = \frac{1}{4}x, y = \frac{1}{x}$. (30 points)

$\frac{y}{x} = 2$ $\quad$ $\frac{y}{x} = \frac{1}{4}$

$xy = u$ $\qquad 1 \le u \le 4$

$\frac{y}{x} = v$ $\qquad \frac{1}{4} \le v \le 2$





$\frac{xy}{y/x} = \frac{u}{v}$

$x^2 = \frac{u}{v} \Rightarrow x = \frac{\sqrt{u}}{\sqrt{v}}$

$xy \cdot \frac{y}{x} = uv$

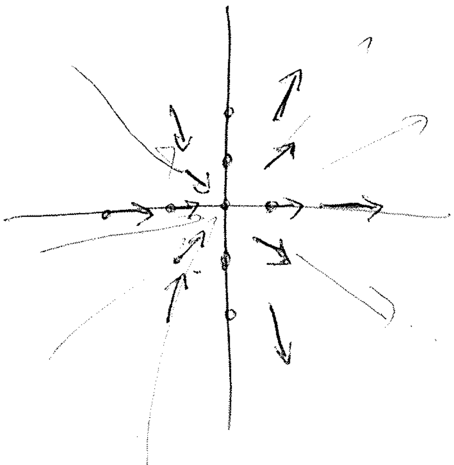
$y^2 = uv$

$y = \sqrt{u}\sqrt{v}$

$$\begin{vmatrix} \frac{1}{2}(uv)^{-1/2} & \frac{1}{2}\left(\frac{v}{u}\right)^{1/2} \\ -\frac{1}{2}\frac{\sqrt{u}}{\sqrt{v^3}} & \frac{1}{2}\left(\frac{u}{v}\right)^{1/2} \end{vmatrix} = \frac{1}{4}\frac{1}{v} + \frac{1}{4}\frac{1}{v} = \frac{1}{2} \cdot \frac{1}{v}$$

$$\frac{1}{2}\int_1^4 \int_{1/4}^2 e^{-\frac{u}{2}} \frac{1}{v}\, dv\, du = \ln 8\left(\frac{1}{\sqrt{e}} - \frac{1}{e^2}\right)$$

4. Plot the 2-dimensional vector field $\vec{F}(x,y) = x^2\vec{i} + xy\vec{j}$. Describe in words the shape of the field. Plot at least 15-20 points. (20 points)



X	Y	F
1	2	<1, 2>
-1	2	<1, -2>
1	-2	<1, -2>
-1	-2	<1, 2>

X	Y	F
0	0	<0, 0>
0	1	<0, 0>
0	-1	<0, 0>
1	0	<1, 0>
-1	0	<1, 0>
1	1	<1, 1>
-1	1	<1, -1>
-1	-1	<1, 1>
1	-1	<1, -1>
0	2	<0, 0>
0	-2	<0, 0>
2	0	<4, 0>
-2	0	<4, 0>

5. Calculate the potential function, if it exists, for the function $\vec{F}(x,y,z) = (2xyz + z^2)\vec{i} + (x^2z - 3)\vec{j} + (x^2y + 2xz)\vec{k}$ or prove that it does not. (15 points)

$\int 2xyz + z^2 \, dx =$

$x^2yz + xz^2 + \text{stuff}$

$\int x^2z - 3 \, dy$

$x^2yz - 3y + \text{stuff}$

$\int x^2y + 2xz \, dz$

$x^2yz + xz^2 + \text{stuff}$

$f(x,y,z) = x^2yz + xz^2 - 3y + K$

6. Evaluate the line integrals in each of the problems below. (15 points each)

a. $\int_C \vec{F}\cdot d\vec{r},\ F(x,y,z) = yz\vec{i} + xz\vec{j} + xy\vec{k},\ C: \textit{line from } (0,0,0) \textit{ to } (5,3,2)$

$$r(t) = 5t\,\hat{i} + 3t\,\hat{j} + 2t\,k \qquad 0 \le t \le 1$$

$$yz = 6t^2 \qquad xz = 10t^2 \qquad xy = 15t^2$$

$$r'(t) = 5\hat{i} + 3\hat{j} + 2\hat{k}$$

$$F(r) = 6t^2\hat{i} + 10t^2\hat{j} + 15t^2 k$$

$$30t^2 + 30t^2 + 30t^2 = 90t^2$$

$$\int_0^1 90t^2\,dt = 30t^3\Big|_0^1 = \boxed{30}$$

or not that this vector function $\vec{F}$ is conservative

$f(x,y,z) = xyz$

$$5\cdot 3\cdot 2 - 0\cdot 0\cdot 0 = \boxed{30}$$

b. $\int_C (3y - x)dx + y^2 dy,\ C: \vec{r}(t) = 2t\vec{i} + 10t\vec{j},\ 0 \le t \le 1$

$$dx = 2dt \qquad dy = 10dt$$

$$\int_0^1 (30t - 2t)\,2dt + 100t^2 \cdot 10dt =$$

$$\int_0^1 \cancel{28t}\ 56t\,dt + 1000t^2\,dt$$

$$28t^2 + \frac{1000}{3}t^3\Big|_0^1 = 28 + \frac{1000}{3} = \boxed{\frac{1084}{3}}$$

7. Find the curl and the divergence of the vector field $F(x,y,z) = x^2 z\vec{i} - 2xz\vec{j} + yz\vec{k}$. (20 points)

$$\text{div. } \nabla\cdot F = 2xz + 0 + y = 2xz + y$$

$$\nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 z & -2xz & +yz \end{vmatrix} = (z + 2x)\hat{i} - (0 - x^2)\hat{j} + (-2z - 0)\hat{k}$$

$$= (z + 2x)\hat{i} + x^2\hat{j} - 2z\hat{k}$$

8. Find the curl of the function $\vec{F}(\rho,\varphi,\theta)=\left\langle \overset{M}{\frac{1}{\rho}}, \overset{N}{\rho^2\sin^3(\varphi\theta)}, \overset{P}{\ln\rho} \right\rangle$ in spherical coordinates using the formula

$$\left\langle \frac{1}{\rho\sin\varphi}\left(\frac{\partial}{\partial\varphi}\underset{\substack{\rho^2\sin\varphi\\ \sin^3\varphi\theta}}{[\sin\varphi N]}-\frac{\partial P}{\partial\theta}\right), \frac{1}{\rho}\cdot\left(\frac{1}{\sin\varphi}\frac{\partial M}{\partial\theta}-\frac{\partial}{\partial\rho}\underset{\rho^3\sin^3(\varphi\theta)}{[\rho N]}\right), \frac{1}{\rho}\cdot\left(\frac{\partial}{\partial\rho}\underset{\rho\ln\rho}{[\rho P]}-\frac{\partial M}{\partial\theta}\right)\right\rangle.$$ (10 points)

$$\frac{1}{\rho\sin\varphi}\left(\rho^2\cos\varphi\sin^3\varphi\theta+\rho^2\cancel{\sin\varphi}\,3\sin^2(\varphi\theta)\cos(\varphi\theta)\cdot\theta-0\cdot\right)$$

$$\boxed{=\left\langle \rho\tan\varphi\sin^3\varphi\theta + 3\rho\theta\sin^2(\varphi\theta)\cos(\varphi\theta),\ 3\rho\sin^3(\varphi\theta),\ \frac{\ln\rho}{\rho}+\frac{1}{\rho}\right\rangle}$$

$$\frac{1}{\rho}\left(0-3\rho^2\sin^3(\varphi\theta)\right)$$

$$\frac{1}{\rho}\left(\ln\rho+\frac{\cancel{\rho}}{\rho}\right)=\frac{\ln\rho}{\rho}-\frac{1}{\rho}-0$$

9. Evaluate the line integral $\int_C [2(x+y)\vec{i}+2(x+y)\vec{j}]\cdot d\vec{r}$; C: smooth curve from (-2,2) to (4,3) using the Fundamental Theorem of Line Integrals. (15 points)

$$f(x,y)=(x+y)^2$$

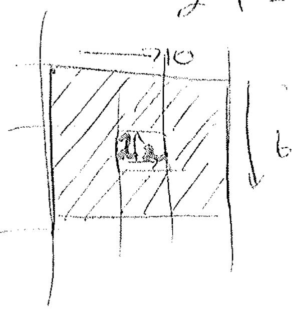
$$(4+3)^2-(-2+2)^2=7^2-0^2=\boxed{49}$$

10. Use Green's Theorem to evaluate the integral for the given path. $\int_C (y-x)dx+(2x-y)dy$; C: boundary of the region lying inside the rectangle bounded by x= -5, x=5, y=-3, and y=3, and outside the square bounded by x= -1, x=1, y=-1, and y=1. [Hint: you may not need to calculate the integral explicitly if you can find the area by another means.] (15 points)

$\frac{\partial N}{\partial x} = 2 \quad \frac{\partial M}{\partial y} = 1$

$\iint_R 1\, dA =$

$2-1 = 1$



$60-4 = \text{area} = 59$

$\boxed{56}$

11. Find the tangent plane and normal line to the graph at the given point $\vec{r}(u,v) = u\vec{i} + v\vec{j} + uv\vec{k}, P(1,1,1)$.(15 points)

$u=1, v=1$

$r_u = \hat{i} + 0\hat{j} + v\hat{k}$

$r_v = 0\hat{i} + \hat{j} + u\hat{k}$

$r_u \times r_v \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = (0-v)\hat{i} - (u-0)\hat{j} + (1)\hat{k}$

$-v\hat{i} - u\hat{j} + \hat{k}$

$-1\hat{i} - 1\hat{j} + 1\hat{k}$

$\langle -1, -1, 1 \rangle$

tangent plane =

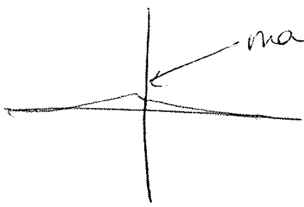
$-1(x-1) - 1(y-1) + 1(z-1) = 0$

line:

$\frac{x-1}{-1} = \frac{y-1}{-1} = \frac{z-1}{1}$

12. Find the point on the curve $y = e^x$ where the curvature K is a maximum, and the limit as x→∞. Find any points where the curvature is zero. (15 points)

$$K = \frac{e^x}{\left(\sqrt{1+e^{2x}}\right)^3}$$



max at $x = -.34657$

Can use calc to find

Curvature $\neq 0$

13. Use spherical coordinates to find the limit of $\lim_{(x,y,z)\to(0,0,0)} \tan^{-1}\left[\frac{1}{x^2+y^2+z^2}\right]$. (10 points)

$$\lim_{(\rho,\varphi,\theta)\to(0,\varphi,\theta)} \tan^{-1}\left(\frac{1}{\rho^2}\right) = \pi/2$$

$\rho^2 \to 0$ $\quad \frac{1}{0} \to \infty \quad \tan^{-1}(\infty) \to \pi/2$

14. Find all critical points, relative extrema, and identify whether these points are a relative maximum, a relative minimum or a saddle point (or cannot be determined) at each critical point.
$f(x,y) = x^2 + 6xy + 10y^2 - 4y + 4$ (20 points)

$f_x = 2x + 6y = 0 \qquad x = -3y \qquad (-6, 2)$

$f_y = 20y + 6x - 4 = 0$

$20y + 6(-3y) - 4 = 0$

$20y - 18y - 4 = 0$

$2y = 4$

$y = 2$

$f_{xx} = 2 \qquad x = -6$

$f_{yy} = 20 \qquad D: 2(20) - 6^2 = 40 - 36 = 4 > 0$

$f_{xy} = 6 \qquad f_{xx} > 0 \quad \cup \qquad \boxed{(-6, 2) \text{ minimum}}$

15. Maximize $w=x^2-y^2$, subject to $x+2y-5=0$ (20 points)

$F = x^2 - y^2 - \lambda x - 2\lambda y + 5\lambda$

$F_x = 2x - \lambda = 0 \qquad \lambda = 2x$

$F_y = -2y - 2\lambda = 0 \qquad -2y = 2\lambda \qquad \lambda = -y$

$2x = -y$

$2x + y = 0$

$x + 2y = 5$

$2x + y = 0$

$x + 2y = 5 \qquad \left(-\frac{5}{3}, \frac{10}{3}, -\frac{25}{3}\right)$

$-4x - 2y = 0$

$-3x = 5$

$x = -\frac{5}{3}$

$y = -2\left(-\frac{5}{3}\right) = +\frac{10}{3}$

16. Find the Jacobian for the three-dimensional change of coordinates: (10 points)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$x = u(1-v), y = uv(1-w), z = uvw$$

$$\begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix} = (1-v)\left(u(1-w)uv + u^2vw\right) + (+u)\left(v(1-w)uw + v^2uw\right) + (0)$$

$$(1-v)\left(u^2v(1-w) + u^2v(w)\right) + u\left(uvw(1-w) + v^2uw\right)$$

$$u^2v(1\cancel{-w}\cancel{+w}) \qquad u^2vw(1-w) + v^2u^2w$$

$$(1-v)(u^2v)$$

$$u^2v - u^2v^2 + u^2vw - u^2vw^2 + v^2u^2w$$

$$u^2(v - v^2 + vw - vw^2 + v^2w)$$

$$\boxed{u^2v(1 - v + w - w^2 + vw)}$$