Name KEY

Math 255, Exam #2, Summer 2012

Instructions: Show all work. Answers with no work will be graded all or nothing unless the point of the problem is to show the work (in which case, no work will receive no credit). Use exact values (fractions and square roots, etc.) unless the problem tells you to round, is a word problem, or begins with decimal values.

1. The general solution to the differential equation $x^2y'' - xy' + y = 0$ is given by $y = c_1x + c_2x\ln x$, which is defined on the interval $(0,\infty)$. Find a member of the family that satisfies the initial conditions y(1)=3, $y'(1) = -1$. (10 points)

$y = c_1x + c_2x\ln x$

$3 = c_1(1) + \cancel{c_2(1)\ln(1)}$

$c_1 = 3$

$y' = c_1 + c_2\ln x + c_2x\cdot\frac{1}{x}$

$c_1 + c_2\ln x + c_2$

$(c_1 + c_2) + c_2\ln x$

$-1 = (3 + c_2) + \cancel{c_2(\ln(1))}$

$-1 - 3 = -4 = c_2$

$y = 3x + (-4)x\ln x$

2. For the same general solution and differential equation in problem #1, solve for the member of the family that satisfies the boundary conditions y(1)=3, and y(2)=15. Other than the values of the constants, is there anything substantially different about these solutions? Explain. (10 points)

$y = c_1x + c_2x\ln x$

$3 = c_1 \cancel{+} \cancel{0}$

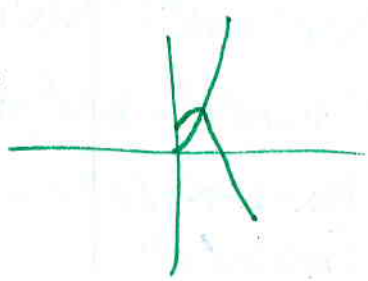
$c_1 = 3$

$15 = 3(2) + c_2(2)\ln 2$

$9 = c_2(2)\ln 2$

$c_2 = \frac{9}{2\ln 2}$

not really...
the constants are more complicated
this one increases rather than decreases & turns around



$y = 3x + \left(\frac{9}{2\ln 2}\right)x\ln x$

3. Use the Wronskian to determine if the following sets of solutions are linearly independent.
   a. $f(x) = \cos(x), g(x) = \cos^2 x$ on all real numbers (10 points)

$$W = \begin{vmatrix} \cos x & \cos^2 x \\ -\sin x & -2\cos x \sin x \end{vmatrix} \quad -2\cos^2 x \sin x + \cos^2 x \sin x$$

$$= -\cos^2 x \sin x$$

$\neq 0$ (except at all multiples of $\pi/2$)

yes, they are linearly independent

   b. $f(x) = x, g(x) = x^{-2}, h(x) = x^{-2}\ln x$ on the interval $(0,\infty)$ (15 points)

$$W = \begin{vmatrix} x & x^{-2} & x^{-2}\ln x \\ 1 & -2x^{-3} & -2x^{-3}\ln x + x^{-3} \\ 0 & 6x^{-4} & 6x^{-4}\ln x - 2x^{-4} + -3x^{-4} \end{vmatrix}$$

$$x\left[(-2x^{-3})(6x^{-4}\ln x - 5x^{-4}) - 6x^{-4}(-2x^{-3}\ln x - x^{-3}\right) - x^{-2}\left[6x^{-4}\ln x - 5x^{-4}\right] +$$

$$x^{-2}\ln x\left[6x^{-4}\right] = x^{-6}(-12\ln x) + \cancel{x^{-6}(-5)} + 12x^{-7}\ln x + 6x^{-7} \cancel{- 6x^{-6}\ln x} \cancel{+ 5x^{-6}}$$

$$\cancel{+6x^{-6}\ln x} \quad \neq 0 \text{ except (undef. at } x=0)$$ linearly independent

4. Use reduction of order to solve the differential equation $(x-1)y'' - xy' + y = 0, x > 1, y_1(x) = e^x$ (20 points)

$y = ve^x$

$y' = v'e^x + ve^x$

$y'' = v''e^x + 2v'e^x + ve^x$

$(x-1)(v''e^x + 2v'e^x + ve^x) - x(v'e^x + ve^x) + ve^x = 0$

$xv''e^x - v''e^x + 2xv'e^x - 2v'e^x \cancel{+ xve^x} \cancel{- ve^x} - xv'e^x \cancel{- xve^x} \cancel{+ ve^x} = 0$

$(x-1)v''e^x + (2x-2-x)v'e^x = 0$

$(x-1)v''e^x + (x-2)v'e^x = 0$

$v'' + \frac{x-2}{x-1}v' = 0$

$v' = w$

$w' + \frac{x-2}{x-1}w = 0 \qquad w' = -\frac{x-2}{x-1}w \Rightarrow$

$w' = -\int \frac{x-2}{x-1}dx = -\int 1 - \frac{1}{x-1}dx = -x + \ln|x-1| = 1$

$$x-1 \overline{)\,x-2} = 1, \quad -x+1, \quad -1$$

$w = e^{-x + \ln|x-1|} = e^{-x}e^{\ln|x-1|} =$

$= e^{-x}(x-1) = v'$

$v = \int (x-1)e^{-x}$ $\quad u = x-1 \quad dv = e^{-x}$, $\quad du = dx \quad v = -e^{-x}$

$= -(x-1)e^{-x} + \int e^{-x} = (x-1)e^{-x} - e^{-x}$

$y_2 = e^x\left[(x-1)e^{-x} - e^{-x}\right] =$

$y_2 = (x-1) - 1 = \boxed{x-2}$

5. Find the given solution of the second order differential equation $6y''-5y'+y=0, y(0)=4, y'(0)=0$. (20 points)

$6r^2-5r+1=0$

$(3r-1)(2r-1)=0$

$r=\frac{1}{3}, r=\frac{1}{2}$

$y_c = Ae^{\frac{1}{3}x}+Be^{\frac{1}{2}x}$

$4 = A+B \qquad A=4-B$

$y_c' = \frac{1}{3}Ae^{\frac{1}{3}x}+\frac{1}{2}Be^{\frac{1}{2}x}$

$0 = \frac{A}{3}+\frac{B}{2} \quad \times 6$

$0 = 2A+3B$

$0 = 2(4-B)+3B$

$8-2B+3B=0$

$8+B=0$

$B=-8$

$A=4-(-8)=12$

$y = 12e^{\frac{1}{3}x}-8e^{\frac{1}{2}x}$

6. Find the solution of the higher order differential equation $y'''+3y''-4y'-12y=0$. (20 points)

$r^3+3r^2-4r-12=0$

$r^2(r+3)-4(r+3)=0$

$(r^2-4)(r+3)=0$

$(r-2)(r+2)(r+3)=0$

$r=2,-2,-3$

$y_c = Ae^{2x}+Be^{-2x}+Ce^{-3x}$

7. For each example in the table below, and given the fundamental solutions for a second-order differential equation, and the forcing term, use this information to determine the best guess for the particular solution you would start with to solve for the non-homogeneous equation. (5 points each)

#	y_1	y_2	g(t)	Ansatz Y(t)
1	e^{-2t}	te^{-2t}	e^{-t}	Ae^{-t}
2	e^{-t}	e^{t}	$\sinh t$ $\frac{1}{2}e^{t}-\frac{1}{2}e^{-t}$	$Ate^{t}+Bte^{-t}$
3	$e^{t}\sin t$	$e^{t}\cos t$	$t^2+e^{t}\cos t$	$At^2+Bt+C+Dt\cos t+Et\sin t$
4	sin(3t)	cos(3t)	3cos(3t)	$At\cos 3t+Bt\sin 3t$

8. Use variation of parameters to solve for the general solution to the differential equation $y''+2y'+y=e^{-t}\ln t$. (25 points)

$g=e^{-t}\ln t$

$y_p=-y_1\int\frac{y_2 g}{W}+y_2\int\frac{y_1 g}{W}$

$r^2+2r+1=0$

$(r+1)^2=0$

$r=-1$ repeated

$y_c=\underset{y_1}{Ae^{-t}}+\underset{y_2}{Bte^{-t}}$

$W=\begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t}-te^{-t}\end{vmatrix}=e^{-2t}\cancel{-te^{-2t}}+\cancel{te^{-2t}}=e^{-2t}$

$y_p=-e^{-t}\int\frac{t\cancel{e^{-t}}\cdot\cancel{e^{-t}}\ln t}{\cancel{e^{-2t}}}+te^{-t}\int\frac{\cancel{e^{-t}}\cancel{e^{-t}}\ln t}{\cancel{e^{-2t}}}=$

$-e^{-t}\int t\ln t\,dt+te^{-t}\int\ln t$

$u=\ln t$ $\quad dv=t$
$du=\frac{1}{t}$ $\quad v=\frac{t^2}{2}$

$u=\ln t$ $\quad dv=dt$
$du=\frac{1}{t}$ $\quad v=t$

$-e^{-t}\left[\frac{t^2}{2}\ln t-\int\frac{t}{2}dt\right]+te^{-t}\left[t\ln t-\int 1dt\right]=$

$-e^{-t}\left[\frac{t^2}{2}\ln t-\frac{t^2}{4}\right]+te^{-t}\left[t\ln t-t\right]=-e^{-t}\frac{t^2}{2}\ln t+t^2e^{-t}\ln t+e^{-t}\frac{t^2}{4}-\frac{t^2}{1}e^{-t}$

$\boxed{y_p=\frac{1}{2}e^{-t}\ln t\cdot t^2+e^{-t}\cdot\frac{3}{4}t^2}$

$\boxed{y=Ae^{-t}+Bte^{-t}+\frac{t^2}{2}e^{-t}\ln t+\frac{3}{4}t^2e^{-t}}$

9. Use the Cauchy-Euler method to solve the differential equation $x^2y'' - 3xy' + 4y = 0, y(1) = 5, y'(1) = 3.$ (20 points)

$x^n = y$

$y' = nx^{n-1}$

$y'' = (n-1)nx^{n-2}$

$x^2(n^2-n)x^{n-2} - 3xnx^{n-1} + 4x^n$

$x^n(n^2-n) - x^n(3n) + x^4(4) = 0$

$x^n[n^2 - 4n + 4] = 0$

$(n-2)^2 = 0$

$n = 2$

$y_1 = x^2 \quad y_2 = \ln x \cdot x^2$

$y' = \underbrace{2x}_{A} + \underbrace{2x\ln x + x^2 \cdot \frac{1}{x}}_{B}$

$5 = A(1)^2 + \cancel{B(\ln(1))\cdot(1)^2}$

$A = 5$

$3 = 2A(1) + \cancel{2(1)(\ln(1))} + (1)B$

$3 = 10 + B$

$B = -7$

$\boxed{y = 5x^2 - 7x^2\ln x}$

10. If an undamped spring-mass system with a mass that weighs 8 lbs and a spring constant 0.5 lbs/in is suddenly set in motion at t=0 by an external force of 3cos(5t) pounds, determine that position of the mass at any time and draw a graph of the displacement versus t. (25 points)

$W = 8$

$m = \frac{8}{32} = \frac{1}{4}$

$k = \frac{1}{2}$

$\frac{1}{4}y'' + \frac{1}{2}y = 3\cos 5t \quad \times 4$

$y(0) = 0$
$y'(0) = 0$

$y'' + 2y = 12\cos 5t$

$r^2 + 2 = 0$

$r = \pm\sqrt{2}\,i$

$y_c = A\cos\sqrt{2}t + B\sin\sqrt{2}t$

$y_p = C\sin 5t + D\cos 5t$

$y_p' = 5C\cos 5t - 5D\sin 5t$

$y_p'' = -25C\sin 5t - 25D\cos 5t$

$-25C\sin 5t - 25D\cos 5t + 2C\sin 5t + 2D\cos 5t = 12\cos 5t$

$-25C + 2C = 0 \quad C = 0$

$-25D + 2D = 12 \quad -23D = 12 \quad D = \frac{-12}{23}$

$y = A\cos\sqrt{2}t + B\sin\sqrt{2}t - \frac{12}{23}\cos 5t$

$0 = A(1) - \frac{12}{23}(1) \quad A = \frac{12}{23}$

$0 = -\sqrt{2}A\sin\sqrt{2}t + \sqrt{2}B\cos\sqrt{2}t + \frac{12\cdot 5}{23}\sin 5t$

$0 = \sqrt{2}B\cos\sqrt{2}t \quad B = 0$

$\boxed{y = \frac{12}{23}\cos\sqrt{2}t - \frac{12}{23}\cos 5t}$

11. A mass weighing 5 lbs stretches a spring 2 inches. If the mass is pushed upwards, contracting the spring a distance of 1 inch and then set in motion with a downward velocity of 4 ft/sec, and if there is no damping, find the position y of the mass at any time t. Determine the frequency, period, amplitude and phase shift of the motion. (25 points)

$W = 5$

$m = \frac{5}{32}$

$5 = k\frac{1}{6}$

$k = 30$

$y(0) = +\frac{1}{12}$

$y'(0) = -4$

$\frac{5}{32}y'' + 30y = 0$

$5y'' + 960y = 0$

$y'' + 192y = 0$

$r^2 + 192 = 0$

$r = \pm\sqrt{192}\,i$

$= 8\sqrt{3}\,i$

$y_c = A\cos 8\sqrt{3}t + B \sin 8\sqrt{3}t$

$\frac{1}{12} = A(1) + \cancel{B(0)}$

$A = \frac{1}{12}$

$y_c' = -8\sqrt{3}\left(\frac{1}{12}\right)\sin 8\sqrt{3}t + 8\sqrt{3}B\cos 8\sqrt{3}t$

$4 = \cancel{-8\sqrt{3}\left(\frac{1}{12}\right)(0)} + 8\sqrt{3}B(1)$

$\frac{-\cancel{4}}{2\cancel{8}\sqrt{3}} = B = \frac{-1}{2\sqrt{3}}$

$\omega = \text{frequency} = 8\sqrt{3}$

$T = \text{period} = \frac{2\pi}{8\sqrt{3}} = \frac{\pi}{4\sqrt{3}}$

amplitude $=$

$\sqrt{\left(\frac{1}{12}\right)^2 + \left(\frac{-1}{2\sqrt{3}}\right)^2} =$

$\sqrt{\frac{1}{144} + \frac{1}{12}} = \sqrt{\frac{1}{144} + \frac{12}{144}} =$

$\boxed{\frac{\sqrt{13}}{12}}$

phase shift $\varphi = \tan^{-1}\frac{A}{B} =$

$\tan^{-1}\frac{\frac{1}{12}}{-\frac{1}{2\sqrt{3}}} = \tan^{-1}\left(\frac{-2\sqrt{3}}{12}\right)$

$-.281$ radians

$\approx -16.1^\circ$ (degrees)

$y = \frac{1}{12}\cos 8\sqrt{3}t - \frac{1}{2\sqrt{3}}\sin 8\sqrt{3}t$