Name KEY

Math 255, Final Exam, Summer 2012

Instructions: Show all work. Answers with no work will be graded all or nothing unless the point of the problem is to show the work (in which case, no work will receive no credit). Use exact values (fractions and square roots, etc.) unless the problem tells you to round, is a word problem, or begins with decimal values.

1. Solve the equation $y'' + xy' + 2y = 0, x_0 = 0$ by using power series centered at the given ordinary point. (30 points)

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad y' = \sum_{n=1}^{\infty} c_n x^{n-1}(n) \qquad y'' = \sum_{n=2}^{\infty} c_n (n)(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} c_n n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} c_n n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

$$\sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} c_n n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

$$\sum_{n=1}^{\infty} c_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} c_n n x^n + \sum_{n=1}^{\infty} 2c_n x^n$$
$$+ c_2(2)(1)x^0 \qquad + 2c_0 x^0 = 0$$

$$\sum_{n=1}^{\infty} x^n\left[c_{n+2}(n+2)(n+1) + c_n n + 2c_n\right] + 2c_2 + 2c_0 = 0 \rightarrow c_2 = -c_0$$

$$c_{n+2} = \frac{c_n\cancel{(n+2)}}{\cancel{(n+2)}(n+1)} = \frac{c_n}{n+1}$$

$c_3 = \frac{c_1}{2}$

$c_5 = \frac{c_3}{4} = \frac{c_1}{6}$ $\qquad c_4 = \frac{c_2}{3} \overset{n=2}{=} \frac{-c_0}{3}$

$c_7 = \frac{c_5}{6} = \frac{c_1}{36}$ $\qquad c_6 = \frac{c_4}{5} \overset{n=4}{=} \frac{-c_0}{15}$

$c_9 = \frac{c_7}{8} = \frac{c_1}{288}$ $\qquad c_8 = \frac{c_6}{7} = \frac{-c_0}{105}$

$$y_1 = c_0\left(1 - x^2 - \tfrac{1}{3}x^4 - \tfrac{1}{15}x^6 - \tfrac{1}{105}x^8 \ldots\right)$$

$$y_2 = c_1\left(x + \tfrac{1}{2}x^3 + \tfrac{1}{6}x^5 + \tfrac{1}{36}x^7 + \tfrac{1}{288}x^9 \ldots\right)$$

2. Define the following terms: (5 points each)

   a. Ordinary point

   A point which satisfies the existence and uniqueness theorem: all $a_n(x)$ are defined, and $a_n(x)$ for the highest order term is not equal to zero.

   b. Singular point

   A point which fails the existence and uniqueness theorem: i.e. one of the conditions above is not true, either a point where $a_n(x)$ is undefined, or where $a_n(x)$ for the highest term is zero.

   c. Regular singular point

   A regular singular point meets the conditions of above, but for an equation in standard form of order n, $(x-x_0)P(x)$ (the coefficient of the n-1 order term) is defined at x0, and $(x-x_0)^2Q(x)$ (the coefficient of the n-2 order term) is defined.

   d. Irregular singular point

   A singular point which is not regular, i.e. it doesn't meet the conditions of above.

   e. State Frobenius' Theorem

   At a regular singular point, a solution will exist if a condition on r can be found for $y = \sum_{n=0}^{\infty} c_n x^{n+r}$.

3. Use the definition of Laplace transforms to find the transform F(s) for the function $f(t) = te^t$. (15 points)

$$\int_0^\infty e^{-st} te^t \, dt = \int_0^\infty e^{t(-s+1)} t \, dt =$$

$$\begin{array}{lll} + & t & e^{-(s-1)t} \\ - & 1 & \dfrac{e^{-(s-1)t}}{-(s-1)} \\ + & 0 & \dfrac{e^{-(s-1)t}}{(s-1)^2} \end{array}$$

$$\cancel{\frac{te^{-(s-1)t}}{-(s-1)}} - \frac{e^{-(s-1)t}}{(s-1)^2}\Bigg|_0^\infty =$$

$$\frac{e^{-(s-1)(0)}}{(s-1)^2} = \boxed{\frac{1}{(s-1)^2}}$$

this agrees w/ short-cut formulas

4. Find the $f(t)=\mathcal{L}^{-1}\left\{\frac{2s-4}{(s^2+s)(s^2+1)}\right\}$ (25 points)

$s(s+1)$

$$\frac{A}{s}+\frac{B}{s+1}+\frac{Cs+D}{s^2+1} \Rightarrow A(s+1)(s^2+1)+B(s)(s^2+1)+(Cs+D)s(s+1) = 2s-4$$

$s=-1$ $\quad B(-1)(2)=2(-1)-4 \Rightarrow -2B=-6 \quad B=3$

$s=0$ $\quad A(1)(1)=-4 \Rightarrow A=-4$

$s=1$ $\quad -4(2)(2)+3(1)(2)+(C+D)(1)(2)=2(1)-4 \Rightarrow$

$\cancel{-16+6} \quad 2C+2D = -2+10=8 \Rightarrow C+D=4$

$s=2$

$-4(3)(5)+3(2)(5)+(2C+D)(2)(3)=0$

$\cancel{-60+30} \quad 2C+D = \frac{+30}{6} \Rightarrow$

$2C+D=5$
$-C-D=-4$

$C=1$

$D=3$

$$\mathcal{L}^{-1}\left\{\frac{-4}{s}+\frac{3}{s+1}+\frac{s}{s^2+1}+\frac{3}{s^2+1}\right\}=\boxed{-4+3e^{-t}+\cos t+3\sin t}$$

5. Use Laplace Transforms to solve the given initial value problem $y'+6y=e^{4t}, y(0)=2$. (25 points)

$$sY(s)\cancel{-2}+6Y(s)=\frac{1}{s-4}+2$$

$$Y(s)(s+6)=\frac{1}{s-4}+2$$

$$Y(s)=\frac{1}{(s-4)(s+6)}+\frac{2}{s+6}$$

$$\frac{A}{s-4}+\frac{B}{s+6} \Rightarrow A(s+6)+B(s-4)=1$$

$s=-6$ $\quad B(-10)=1 \quad B=\frac{-1}{10}$

$s=4$ $\quad A(10)=1 \quad A=\frac{1}{10}$

$$Y(s)=\frac{\left(\frac{1}{10}\right)}{s-4}+\frac{\left(-\frac{1}{10}\right)+2}{s+6}$$

$$\boxed{y(t)=\frac{1}{10}e^{4t}+\frac{19}{10}e^{-6t}}$$

6. For the following differential equations, determine the order of the equation, whether it is linear or non-linear, and whether it is ordinary or partial. (5 points each)

   a. $\ddot{y} + ty = 0$ 2nd order, ordinary, linear

   b. $\frac{d^2y}{dt^2} + t\frac{dy}{dt} + \sin 2y = e^t$ Second order, ordinary, nonlinear

   c. $y''' + \frac{1}{t}y' + y = t^3$ 3rd order, ordinary, linear

   d. $\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = 0$ 2nd order, partial, nonlinear

   e. $u_{xx} + u_{yy} = u_t$ 2nd order, partial, linear

7. Verify that $y = x^4$ is a solution to the differential equation $x^2y'' - 7xy' + 16y = 0$. (15 points).

$y' = 4x^3 \quad y'' = 12x^2$

$x^2(12x^2) - 7x(4x^3) + 16(x^4) =$

$12x^4 - 28x^4 + 16x^4 = 0$ ✓

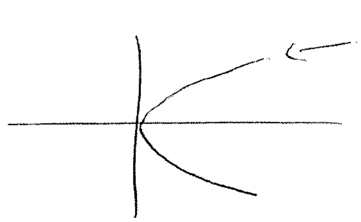
8. Find the region in the x-y plane where the existence and uniqueness theorem guarantees a solution to the equation $(x - y^2)y'\cancel{'} = 16$. Sketch the region. (15 points)

Single prime only.

$$y' = \frac{16}{x-y^2} = f(y) \quad x \neq y^2$$

$$\frac{d}{dy}\left(16(x-y^2)^{-1}\right) = 16(-1)(x-y^2)^{-2}(-2y) = \frac{32y}{(x-y^2)^2}$$

$$x \neq y^2$$



← not defined at $x = y^2$

defined & exists everywhere else

1. For the following equations, determine the solution method (integrating factor/linear, separable equations, homogeneous (substitution y=vx), Bernoulli equations, exact equation) or if the problem must be done numerically. Do not solve the equations. (6 points each)
   a. $y' + 2xy = x^2y^{-4}$

      Benoulli

   b. $y' + y = xe^{-x} + 1$ lenear/integrating factor

   c. $(2xy^2 + 2y + 2x)dx + (2x^2y + 2x - 9y^2)dy = 0$ exact

   d. $xyy' = (y^2 - 1)^{1/2}$ separable

   e. $-y^2dx + x(x + y)dy = 0$ homogeneous

   f. $(x^2 + y^3)y' = (x + 1)(x + y)^2$ numerically

9. Solve the linear differential equation $x^2y' + xy = x^3$ either by integrating factor or by variation of parameters (integral formula). (30 points)

$$y' + \frac{1}{x}y = x$$

$$\mu = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$y = \frac{1}{\mu}\int \mu x \, dx$$

$$y = \frac{1}{x}\int x^2 dx = \frac{\left(\frac{1}{3}x^3\right) + C}{x}$$

$$y = \frac{1}{3}x^2 + \frac{C}{x}$$

$$xy' + y = x^2$$

$$(xy)' = \int x^2 dx =$$

$$\frac{xy}{x} = \frac{\frac{1}{3}x^3 + C}{x}$$

$$\boxed{y = \frac{1}{3}x^2 + \frac{C}{x}}$$

10. Use the Wronskian to determine if the following set of solutions is linearly independent: $g(x) = x^{-2}, h(x) = x^{-2}\ln x$ on the interval (0,∞) (15 points)

$$\begin{vmatrix} x^{-2} & x^{-2}\ln x \\ -2x^{-3} & -2x^{-3}\ln x + \cancel{x^{-2}\cdot x^{-1}} \\ & x^{-3} \end{vmatrix} = \cancel{-2x^{-5}\ln x} + x^{-5} \cancel{- (-2x^{-5})\ln x}$$

$$= x^{-5}$$

linearly independent for all $x \neq 0$

11. Find the given solution of the second order differential equation $6y''-5y'+y=0, y(0)=4, y'(0)=0$. (25 points)

$6r^2-5r+1=0$

$(3r-1)(2r-1)=0$

$r=\frac{1}{3} \quad r=\frac{1}{2}$

$y=Ae^{\frac{1}{3}t}+Be^{\frac{1}{2}t}$

$4=Ae^{0}+Be^{0} \qquad 4=A+B \quad \Rightarrow \quad -2\cancel{A}-2B=-8$

$y'=\frac{1}{3}Ae^{\frac{1}{3}t}+\frac{1}{2}Be^{\frac{1}{2}t} \qquad 0=2A+3B \qquad \cancel{2A}+3B=0$

$B=-8$

$0=\frac{1}{3}A+\frac{1}{2}B$

$4 \pm A-8$

$A=12$

$$\boxed{y=12e^{\frac{1}{3}t}+(-8)e^{\frac{1}{2}t}}$$

12. For each example in the table below, and given the fundamental solutions for a second-order differential equation, and the forcing term, use this information to determine the best guess for the particular solution you would start with to solve for the non-homogeneous equation. (6 points each)

#	y_1	y_2	g(t)	Ansatz Y(t)
1	e^{-2t}	te^{-2t}	e^{-t}	Ae^{-t}
2	e^{-t}	e^{t}	$\cosh t$ $\frac{1}{2}(e^{t}+e^{-t})$	$Ate^{t}+Bte^{-t}$ or $At\cosh t+Bt\sinh t$
3	$e^{t}\sin t$	$e^{t}\cos t$	$t^2+e^{t}\sin t$	$At^2+Bt+C+Dte^{t}\sin t+Ete^{t}\cos t$
4	sin(3t)	cos(3t)	3cos(2t)	$A\sin 2t+B\cos 2t$

13. A mass weighing 5 lbs stretches a spring 2 inches. If the mass is pushed upwards, contracting the spring a distance of 1 inch and then set in motion with a downward velocity of 4 ft/sec, and if there is no damping, find the position y of the mass at any time t. Determine the frequency, period, amplitude and phase shift of the motion. (30 points)

$F = 5\text{lbs}$ $\quad 2\text{in} = \frac{1}{6}\text{ft}$

$5 = k\left(\frac{1}{6}\right)$

$k = 30$

$M = \frac{F}{g} = \frac{5}{32}$

$y(0) = \frac{1}{12}\text{ft}$

$y'(0) = -4$

$My'' + \gamma y' + ky = F(t)$

$\gamma = 0$

$\frac{32}{5} \cdot \frac{5}{32} y'' + \frac{32}{5} 30 y = 0$

$y'' + 192 y = 0$

$r^2 + 192 = 0 \Rightarrow r^2 = -192$

$r = \pm\sqrt{192}i = \pm 2\sqrt{48}i = \pm 4\sqrt{12}i$

$= \pm 8\sqrt{3}i$

$y = A\sin(8\sqrt{3}t) + B\cos(8\sqrt{3}t)$

$\frac{1}{12} = \cancel{A\sin(0)} + B\cancel{\cos(0)} \Rightarrow \frac{1}{12} = B$

$y' = 8\sqrt{3}A\cos(8\sqrt{3}t) + -8\sqrt{3}B\sin(8\sqrt{3}t)$

$-4 = 8\sqrt{3}A\cancel{\cos(0)} - \cancel{8\sqrt{3}\left(\frac{1}{12}\right)\sin(0)}$

$\frac{-\cancel{4}}{{}_2\cancel{8}\sqrt{3}} = A = \frac{-1}{2\sqrt{3}}$

$$\boxed{y = \frac{-1}{2\sqrt{3}}\sin(8\sqrt{3}t) + \frac{1}{12}\cos(8\sqrt{3}t)}$$

frequency: $8\sqrt{3}$

period $= \frac{2\pi}{8\sqrt{3}} = \frac{\pi}{4\sqrt{3}}$

amplitude $=$

$\sqrt{\left(\frac{1}{12}\right)^2 + \left(\frac{1}{2\sqrt{3}}\right)^2} =$

$\sqrt{\frac{1}{144} + \frac{1}{12}} =$

$\sqrt{\frac{13}{144}} = \frac{\sqrt{13}}{12}$

phase shift $= \tan^{-1}\left(\frac{-\frac{1}{12}}{\frac{1}{2\sqrt{3}}}\right)$

$= -.291$ radians [illegible]