Name KEY

Math 2568, Exam #3 – Part 1, Summer 2013

Instructions: On this portion of the exam, you may **NOT** use a calculator. Show all work. Answers must be supported by work to receive full credit.

1. Find the eigenvalues and eigenvectors of the matrices below. Be sure to clearly indicate the characteristic equation, and which eigenvalues and eigenvectors go together. (12 points)

   a. $A = \begin{bmatrix} -3 & 7 \\ 5 & -1 \end{bmatrix}$

$(-3-\lambda)(-1-\lambda)-35 = 3+3\lambda+\lambda+\lambda^2-35 =$

$\lambda^2+4\lambda-32=0$

$(\lambda+8)(\lambda-4)=0 \quad \lambda_1=-8, \quad \lambda_2=4$

$\begin{bmatrix} -3-(-8) & 7 \\ 5 & -1-(-8) \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 5 & 7 \end{bmatrix} \quad 5x_1=-7x_2 \Rightarrow \begin{matrix} x_1 = -\frac{7}{5}x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_1 = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$

$\begin{bmatrix} -7 & 7 \\ 5 & -5 \end{bmatrix} \quad \begin{matrix} -7x_1 = -7x_2 \\ x_1 = x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   b. $B = \begin{bmatrix} -4 & 5 \\ -5 & -4 \end{bmatrix}$

$(-4-\lambda)(-4-\lambda)+25 = 16+8\lambda+\lambda^2+25 = \lambda^2+8\lambda+41=0$

$\lambda = \dfrac{-8 \pm \sqrt{64-4(41)}}{2} = \dfrac{-8\pm\sqrt{-100}}{2} = \dfrac{-8\pm 10i}{2} = -4\pm 5i$

matrix already in scaled rotation form (i.e. B=C) similarity P = I

$\begin{bmatrix} -4-(-4+5i) & 5 \\ -5 & -4-(-4+5i) \end{bmatrix} = \begin{bmatrix} -5i & 5 \\ -5 & -5i \end{bmatrix} \quad \dfrac{-5x_1}{-5} = \dfrac{5i\,x_2}{-5} \quad x_2 = x_2$

$\vec{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$

$\vec{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$

2. For the matrix $A = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$, with eigenvalues $\lambda_1 = 3, \lambda_2 = 2$ and eigenvectors $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$, find a similarity transformation matrix P so that A can be diagonalized. Clearly state both P and D. (6 points)

$$P = \begin{bmatrix} 1 & 4 \\ 0 & -3 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

3. Suppose matrix A is a 8x6 matrix with 4 pivot columns. Determine the following. (12 points)

dim Col A = 4   dim Nul A = 2

dim Row A = 4   If Col A is a subspace of R^m, then m = 8

Rank A = 4   If Nul A is a subspace of R^n, then n = 6

4. Given the vectors $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} -4 \\ 4 \\ 4 \end{bmatrix}$ find the following.

a. $\vec{v} \cdot \vec{u}$ (3 points)

$$(-4)(1) + (4)(-1) + (4)(2) = -4 - 4 + 8 = 0$$

b. $\|\vec{u}\|$. (3 points)

$$\sqrt{1^2+(-1)^2+2^2} = \sqrt{1+1+4} = \sqrt{6}$$

c. A unit vector in the direction of $\vec{u}$. (3 points)

$$\begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

d. Find the distance between $\vec{u}$ and $\vec{v}$. (3 points)

$$\vec{u}-\vec{v} \begin{bmatrix} 1-(-4) \\ -1-4 \\ 2-4 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \\ -2 \end{bmatrix} \qquad \|\vec{u}-\vec{v}\| = \sqrt{5^2+(-5)^2+(-2)^2} =$$

$$\sqrt{25+25+4} = \sqrt{54}$$

$3\sqrt{6}$

e. Are $\vec{u}$ and $\vec{v}$ orthogonal? Why or why not? (3 points)

yes. dot product is zero.

5. Consider the stochastic Markov chain matrix given by the matrix $A = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}$. Calculate the equilibrium vector of the system. (6 points)

$P - I$

$$\begin{bmatrix} -.4 & .3 \\ .4 & -.3 \end{bmatrix}$$

$$-.4x_1 = -.3x_2$$

$$x_1 = \frac{\cancel{-}.3}{\cancel{-}.4} x_2$$

$$x_2 = x_2$$

$$\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad 3+4=7$$

$$\vec{q} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$$

6. List at least 8 properties of Invertible Matrices from the Invertible Matrix Theorem. (8 points)

For example:

1) A is invertible
2) $\det A \neq 0$
3) $\dim \text{Nul } A = 0$
4) rank $A = n$
5) $\text{Nul } A = \{\vec{0}\}$
6) reduces to nxn identity
7) transformation is one-to-one
8) matrix C exists such that $CA = I$

etc.

7. Determine if each statement is True or False. (2 points each)

a.	T	F	Two eigenvectors corresponding to distinct eigenvalues are always linearly independent.
b.	T	F	An nxn matrix will always have exactly n eigenvalues.
c.	T	F	If A and B are row equivalent, then their row spaces are the same.
d.	T	F	$P_{C \leftarrow B} = (P_{B \leftarrow C})^{-1}$
e.	T	F	A linearly independent set in a subspace H is a basis for H.
f.	T	F	The equilibrium vector for a stochastic matrix (containing no 1s or 0s) is always unique.
g.	T	F	A matrix is not invertible if and only if 0 is not an eigenvalue of A.
h.	T	F	The eigenvalues of a matrix are on its main diagonal.
i.	T	F	The eigenspace of an nxn matrix with n distinct eigenvalues always form a basis for R^n.
j.	T	F	A trajectory of a dynamical system is a set of ordered vectors $\overrightarrow{x_k}$ that tracks the population values of a system over time.
k.	T	F	The elementary row operations of A do not change its eigenvalues.
l.	T	F	If A is diagonalizable, then A is invertible.
m.	T	F	The complex eigenvalues of a discrete dynamical system either both attract to the origin or both repel from the origin.

Name ______________________________

Math 2568, Exam #3 – Part 2, Summer 2013

Instructions: On this portion of the exam, you *may* use a calculator to perform elementary matrix operations. Support your answers with work (reproduce the reduced matrices from your calculator) or other justification for full credit.

1. a. For the matrix $B = \begin{bmatrix} 19 & 10 \\ -2 & 23 \end{bmatrix}$, with eigenvalues $\lambda = 21 \pm 4i$, with eigenvectors $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \mp 2 \\ 0 \end{bmatrix} i$.
Find one similarity transformation P that will transform B=PCP^{-1}, where C is a scaled rotation matrix. State both P and C. (6 points)

$P = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ $\quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $\quad \lambda = a - bi$

$a = 21 \quad b = 4$

$C = \begin{bmatrix} 21 & -4 \\ 4 & 21 \end{bmatrix}$

b. Use the C matrix from part a, and find the scaling factor and then calculate the angle of rotation of the matrix. Round your angle to 2 decimal places in radians, or to the nearest whole degree. (6 points)

$C = \begin{bmatrix} 21 & -4 \\ 4 & 21 \end{bmatrix}$ $\quad r = \sqrt{21^2 + 4^2} = \sqrt{441 + 16} = \sqrt{457}$

$\sqrt{457} \begin{bmatrix} 21/\sqrt{457} & -4/\sqrt{457} \\ 4/\sqrt{457} & 21/457 \end{bmatrix}$

$\theta = \tan^{-1}\left(\frac{4}{21}\right) \approx 10.78° \approx .188$ radians

$11°$ $\quad$ $.19$ rads

2. Assume that $A = \begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 2 & 4 & -5 & 1 & 2 \\ 1 & 2 & 0 & 3 & 1 \\ 3 & 6 & -1 & 8 & 1 \end{bmatrix}$.

$$\text{rref} \Rightarrow \begin{array}{c} \\ \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} & \text{free} & & \text{free} & \end{array} \end{array}$$

a. Find a basis for the column space of A. (5 points)

$$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

b. Find a basis for the row space of A. (5 points)

$$\text{Row } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Or use first 3 rows of A.

c. Find a basis for the null space of A. (8 points)

$x_1 + 2x_2 + 3x_4 = 0$

$x_2 = \text{free}$

$x_3 + x_4 = 0$

$x_4 = \text{free}$

$x_5 = 0$

$x_1 = -2x_2 - 3x_4$

$x_2 = x_2$

$x_3 = -x_4$

$x_4 = x_4$

$x_5 = 0$

$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

3. Given the bases $B = \{1 + t - 2t^2, 1 + 4t - 5t^2, 1 - t + 4t^2\}$ and $C = \{2 - t + 2t^2, 1 - 2t + 3t^2, t + 2t^2\}$ below, find the change of basis matrices $\underset{C \leftarrow B}{P}$ and $\underset{B \leftarrow C}{P}$. For the B-coordinate vector $\vec{p}$ given as $\left[\vec{p}\right]_B = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$, find the C-coordinate vector for $\vec{p}$, and find the original $p(t)$ in the standard basis. (8 points)

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right\} \quad P_B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & -1 \\ -2 & -5 & 4 \end{bmatrix}$$

$$C = \left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\} \quad P_C = \begin{bmatrix} 2 & 1 & 0 \\ -1 & -2 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

$$P_{C \leftarrow B} = P_C^{-1} P_B = \begin{bmatrix} 11/10 & 2 & 1/10 \\ -6/5 & -3 & 4/5 \\ -3/10 & 0 & 7/10 \end{bmatrix} \quad P_{B \leftarrow C} = \left(P_{C \leftarrow B}\right)^{-1} = \begin{bmatrix} 7/4 & 7/6 & -19/12 \\ -1/2 & -2/3 & 5/6 \\ 3/4 & 1/2 & 3/4 \end{bmatrix}$$

$$P_{C \leftarrow B} [\vec{p}]_B = [\vec{p}]_C = \begin{bmatrix} 47/5 \\ -54/5 \\ 9/5 \end{bmatrix} \qquad P_C [\vec{p}]_C = \begin{bmatrix} 8 \\ 14 \\ -10 \end{bmatrix}$$

$$\boxed{p(t) = 8 + 14t - 10t^2}$$

4. Consider the discrete dynamical system given by the matrix $A = \begin{bmatrix} .5 & .3 \\ -.22 & 1.25 \end{bmatrix}$.

a. Determine the behaviour of the origin for this system: is it a repeller, an attractor or a saddle point? (7 points)

$$(.5 - \lambda)(1.25 - \lambda) + (.22)(.3) = 0$$

$$\lambda = .6018 \qquad \lambda = 1.148$$

Saddle point

one $|\lambda| > 1$, one $|\lambda| < 1$

$$\begin{bmatrix} -.1013 & .3 \\ -.22 & .6482 \end{bmatrix} \quad x_1 = \frac{-.6482}{-.22} x_2 \quad \begin{bmatrix} 2.95 \\ 1 \end{bmatrix}$$

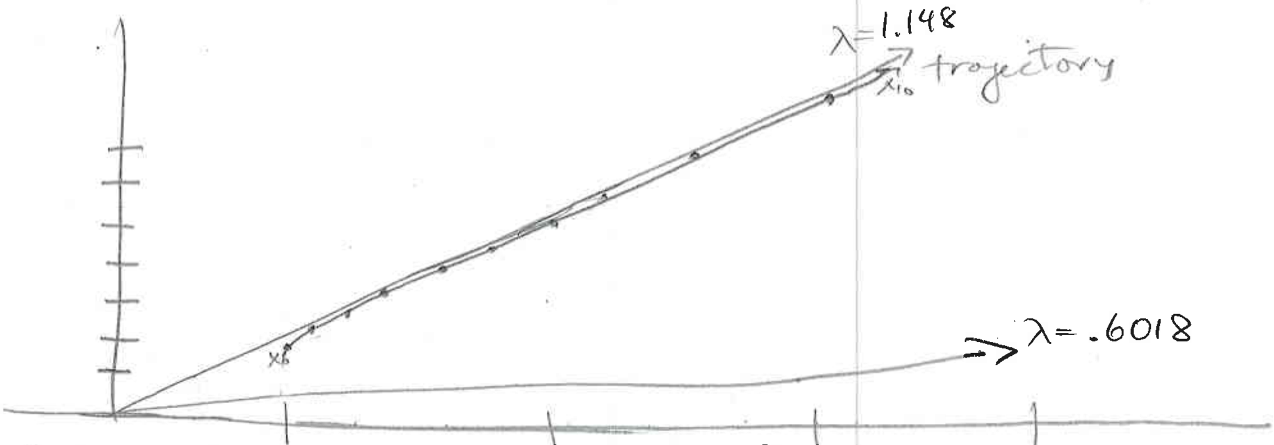
$$\begin{bmatrix} -.648 & .3 \\ -.22 & .102 \end{bmatrix} \quad x_1 = \frac{-.102}{-.22} x_2 \quad \begin{bmatrix} .46 \\ 1 \end{bmatrix}$$

b. Given the initial condition of the population as $x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find 10 points of the trajectory for the system and list them here. Plot the points on a graph together with the eigenvectors of the system. Make sure your graph is big enough to clearly read it. Connect the trajectory with a curve and an arrow indicating the flow of time. (10 points)

$X_1 = \begin{bmatrix} 1.1 \\ 2.28 \end{bmatrix}, X_2 = \begin{bmatrix} 1.234 \\ 2.608 \end{bmatrix}, X_3 = \begin{bmatrix} 1.3994 \\ 2.98852 \end{bmatrix}, X_4 = \begin{bmatrix} 1.596256 \\ 3.427782 \end{bmatrix}$

$X_5 = \begin{bmatrix} 1.82\ldots \\ 3.93 \end{bmatrix}, X_6 = \begin{bmatrix} 2.09\ldots \\ 4.5\ldots \end{bmatrix}, X_7 = \begin{bmatrix} 2.40\ldots \\ 5.18\ldots \end{bmatrix}, X_8 = \begin{bmatrix} 2.75\ldots \\ 5.95\ldots \end{bmatrix}$

$X_9 = \begin{bmatrix} 3.16\ldots \\ 6.83\ldots \end{bmatrix}, X_{10} = \begin{bmatrix} 3.63\ldots \\ 7.84\ldots \end{bmatrix}$



5. Determine if the polynomials $f(x) = \sin x, g(x) = \sin 2x$ are orthogonal under the inner product $< f, g > = \int_{-\pi}^{\pi} f(x)g(x)dx$. [Hint: You may find it helpful to have the identity $\sin(2x) = 2\sin(x)\cos(x)$ to complete the integration. You may use your calculator to check your work, but you should show more than just an answer.] (6 points)

$\sin 2x = 2\sin x \cos x$

$\int_{-\pi}^{\pi} (\sin x)(\sin 2x)\,dx = \int_{-\pi}^{\pi} \sin x\,(2 \sin x \cos x)\,dx =$

$2\int_{-\pi}^{\pi} \sin^2 x \cos x\,dx$ $\quad u = \sin x \quad du = \cos x \quad 2\int u^2\,du = 2\frac{u^3}{3}\Big|$

$\frac{2}{3}\sin^3 x \Big|_{-\pi}^{\pi} = \frac{2}{3}\left[0 - 0\right] = 0$

yes, they are orthogonal