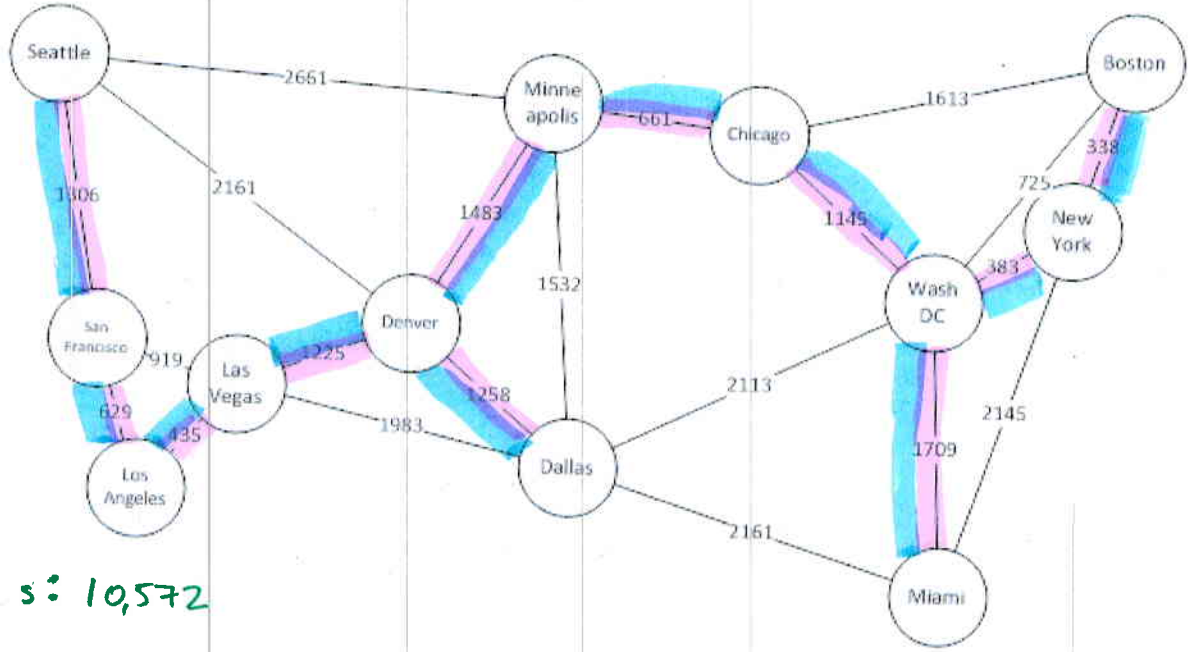
Math 2366, Final Exam, Summer 2014 Name KEY

Instructions: Show all work. Justify answers as completely as possible. If you are asked to prove something, mere computation is not enough. You must explain your reasoning. Be sure to state your conclusion clearly. Incomplete work or justification will not receive full credit. Use exact answers unless specifically asked to round.

1. Find the Minimum Spanning Tree of the graph below using a) Kruskal's Algorithm, b) Prim's Algorithm incident with New York. Do they produce the same result (in terms of the tree structure or the associated cost)? Explain. (16 points)



Kruskal's: 10,572

Prim's: 10,572

the algorithms produced the same tree (MST) here, but edges were added in a different order.

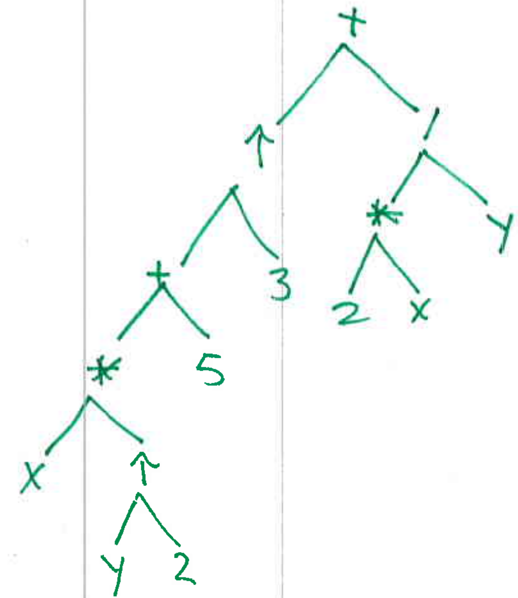
2. Write the mathematical expression $(xy^2+5)^3+\left(\frac{2x}{y}\right)$ using a rooted tree (space for the tree is on the next page). Then express it in the specified notation. (12 points)

a. Infix notation ((x*(y↑2)+5)↑3)+((2*x)/y)

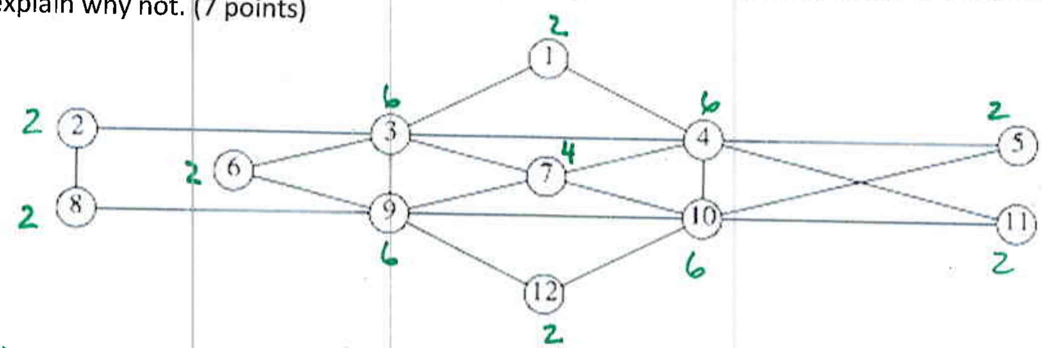
b. Prefix notation + ↑ + * x ↑ y 2 5 3 / * 2 x y

c. Postfix notation x y 2 ↑ * 5 + 3 ↑ 2 x * y / +

Place the tree for Problem #2 here.



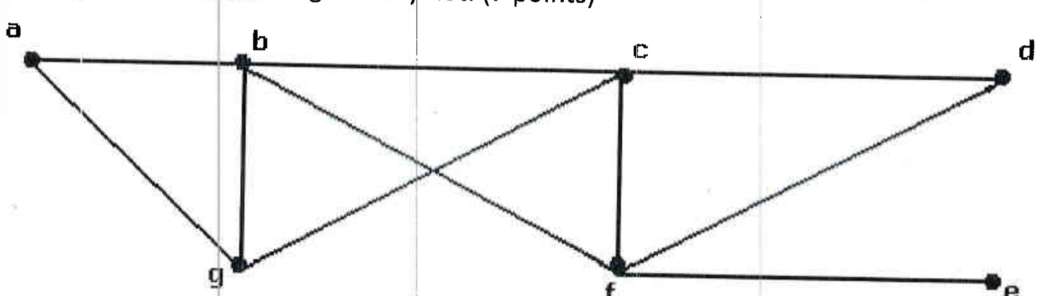
3. Determine if the graph below has an Euler circuit (or in the absence of an Euler circuit, does it have an Euler path?). If it does, find one by listing the vertices used in order. If it does not, explain why not. (7 points)



it has an Euler circuit

answers will vary → 1, 3, 6, 9, 3, 2, 8, 9, 7, 3, 4, 7, 10, 12, 9, 10, 11, 4, 10, 5, 4, 1

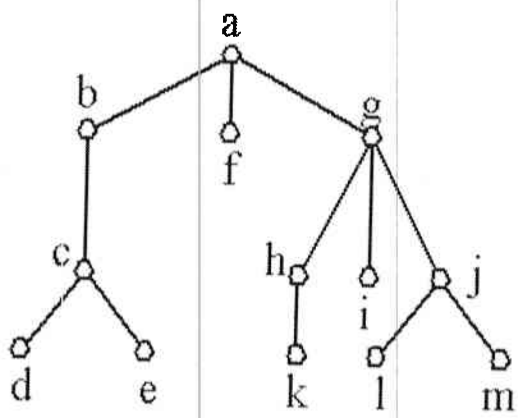
4. Determine if the graph below has a Hamilton circuit (or in the absence of a Hamilton circuit, does it have a Hamilton path?). If it does, find one by listing the vertices used in order. If it does not explain your reasoning for why not. (7 points)



has a Hamilton Path, but no circuit

e, f, d, c, g, b, a answers will vary but must start or stop at e

5. For the tree shown below, find the following vertices corresponding to each description. (2 points each)
   a. The root A
   b. The sibling(s) of B F, G
   c. The parent of M J
   d. The child(ren) of J L, M
   e. All the ancestors of H G, A
   f. All the descendants of C D, E
   g. All the leaves of the tree D, E, F, K, L, M, I



6. For the relation $R = \{(1,1), (1,2), (2,1), (2,2), (2,3), (3,2), (3,3), (3,4), (4,3), (4,4)\}$, on the set $A = \{1,2,3,4\}$, determine if it is reflexive, symmetric, antisymmetric and/or transitive. Explain your reasoning. (8 points)

reflexive since all (a,a) in set

symmetric since all (a,b) ⇒ (b,a)

not antisymmetric since it is symmetric w/ a≠b

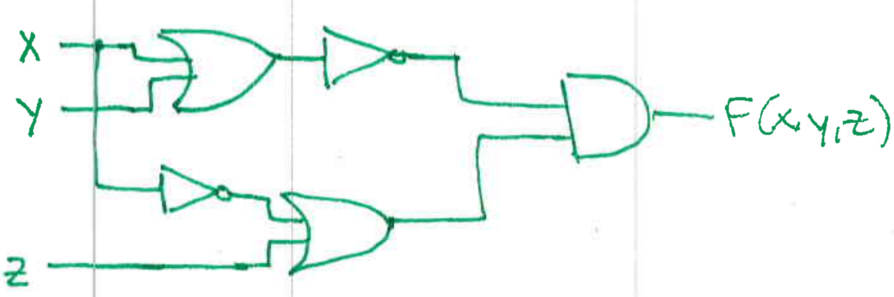
not transitive since (1,2), (2,3) but not (1,3)

7. Consider the relation $R = \{(a, b) | a \text{ is older than } b\}$. Determine if this relation is an equivalence relation. Explain your reasoning. (7 points)

it is not since it's not reflexive a is not older than him- or herself

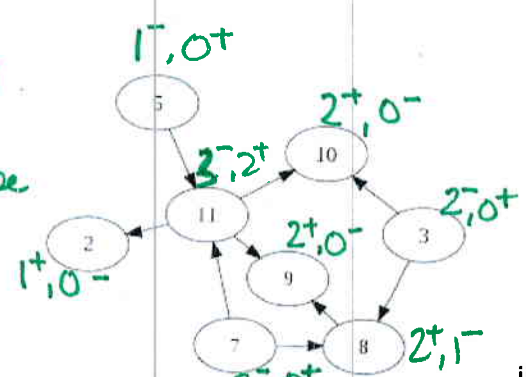
8. Translate the logical proposition $\neg(x \vee y) \wedge (z \vee \neg x)$ into Boolean Algebra. Then draw the logical circuit that represents the expression. (15 points)

$\overline{(x+y)}(z+\bar{x})$

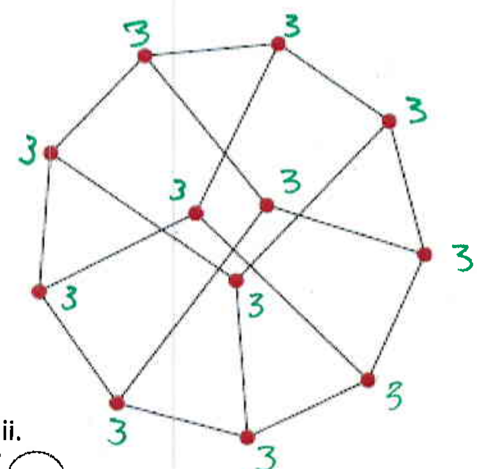


9. For each of the graphs below, determine whether the graphs are a) directed, undirected, or mixed, b) a simple graph, a multigraph or a pseudograph, c) the degree of each vertex (for directed graphs, find the degree-in and the degree-out; you will need to label each vertex if they are not already labeled), d) if the graph is complete, bipartite, a wheel, a cycle, or none of these. (8 points each)

directed
Simple
None of these

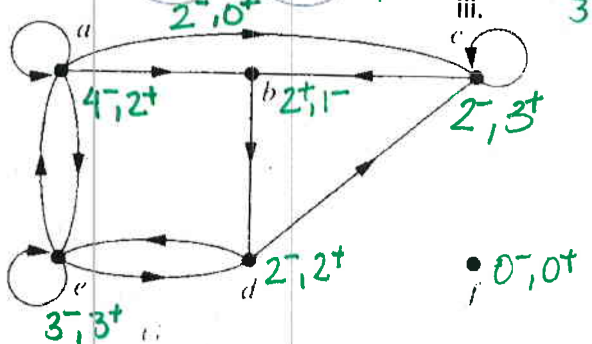


i.



Simple
undirected
none of these

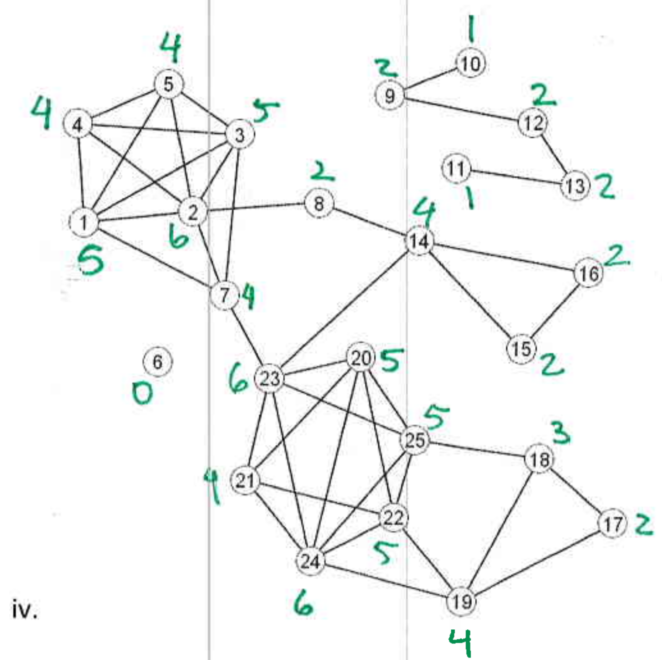
iii.



ii.

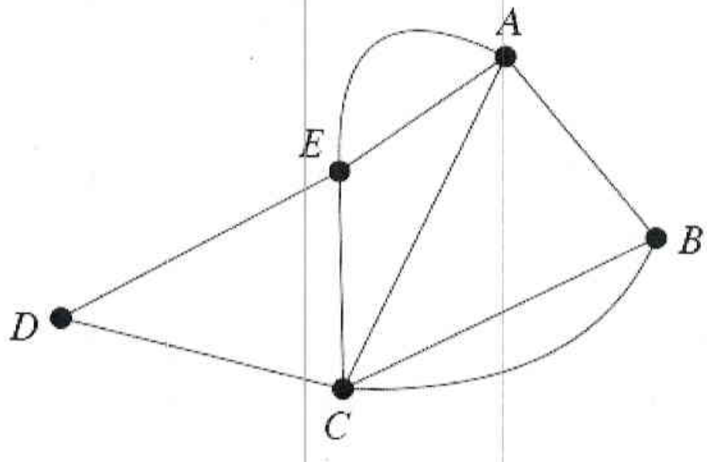
pseudograph
directed
none of these

iv.



Simple
undirected
none of these

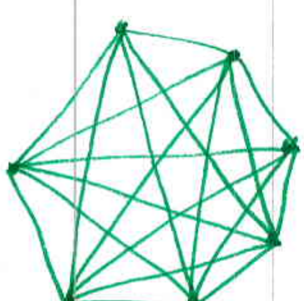
10. For the graph shown below, write an adjacency matrix. (8 points)



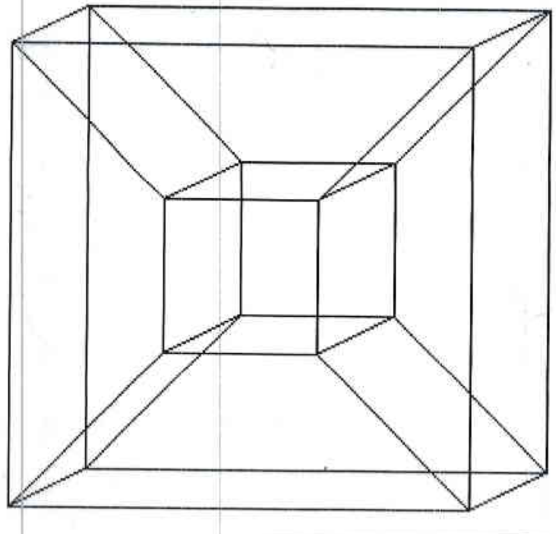
$$\begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 \end{bmatrix}$$

11. Draw examples of graphs of each of the following types. (6 points)

a. K_7



b. Q_4



12. Use mathematical induction to prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. (12 points)

base case $\quad \sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$ ✓

Suppose true for n & show true for n+1

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + n+1 = \frac{n(n+1)}{2} + n+1 = \frac{n(n+1)+2(n+1)}{2} =$$

$$\frac{(n+1)(n+2)}{2} = \frac{(n+1)[(n+1)+1]}{2}$$

which is the formula w/ n+1 plugged in for n.

QED.

13. How many strings of 21 alphanumeric characters are possible if upper and lowercase are different, if there has to be at least one number, and it has to start with an uppercase letter? (8 points)

$\underbrace{26}_{1st} \cdot \underbrace{10 \cdot 20}_{\#+\text{ordering}} (62)^{19} \approx 5.91 \times 10^{37}$

14. How many positive integers not exceeding 1000 are divisible by either 5 or 8? (8 points)

$200 + 125 - 25 = 300$

15. Show that if there are 60 students in a class, there are at least two of them born in the same state. (6 points)

Since there are 50 states, worst case scenario is that the first 50 students will all be from different states, but the next person after that (assuming they are born in the U.S.) will have to be born in the same state as another student.

16. A bagel shop has onion, everything, egg, raisin, plain, poppy seed, salted, rye, chocolate chip, cinnamon swirl, asiago cheese, pizza, pumpernickel swirl and cranberry bagels. How many ways are there to choose a six bagels? (8 points)

$\binom{14+6-1}{6} = \binom{19}{6} = 27,132$

17. How many strings can be made from the letters in the words ACADEMIC REGALIA? (6 points)

A – 4
C – 2
D – 1
E – 2
M – 1
I – 2
R – 1
G – 1
L – 1

$\frac{15!}{4!\,2!\,2!\,2!} = 6,810,804,000$

18. Find the next seven elements in lexicographic order for the permutation 5673142. (12 points)

5673214
5673241
5673412
5673421
5674123
5674132
5674213

19. What is the probability that a five-card poker hand contains no face cards (i.e. it has no Jacks, Queens or Kings)? (8 points)

$$\frac{\binom{40}{5}}{\binom{52}{5}} = .25318\ldots$$

20. What is the probability that a fair die is rolled 25 times and there are more than 5 twos? (8 points)

$1 - P(0 \text{ twos}) - P(1 \text{ two}) - P(2 \text{ twos}) - P(3 \text{ twos}) - P(4 \text{ twos}) - P(5 \text{ twos})$

$$1 - \binom{25}{0}(1/6)^0(5/6)^{25} - \binom{25}{1}(1/6)^1(5/6)^{24} - \binom{25}{2}(1/6)^2(5/6)^{23} - \binom{25}{3}(1/6)^3(5/6)^{22}$$
$$- \binom{25}{4}(1/6)^4(5/6)^{21} - \binom{25}{5}(1/6)^5(5/6)^{20}$$

or $1 - \text{binomialcdf}(25, 1/6, 5) = .22804$

21. Suppose that you flip an unfair coin (with a probability of heads being 3/5). What is the expected number of tails in 5000 flips? (5 points)

$\frac{3}{5} \cdot 5000 = 3000$ heads

& 2000 tails

Since p(tails) = 2/5

$2/5 * 5000 = 2000$

22. Based on the table below, determine if the two variables are independent. If they are, explain how you know. If they are not, explain why not. (18 points)

$p(x,y)$	0	1	2	
0	$\frac{1}{12}$	$\frac{1}{20}$	$\frac{7}{60}$	$1/4$
1	$\frac{2}{9}$	$\frac{2}{15}$	$\frac{14}{45}$	$2/3$
2	$\frac{1}{36}$	$\frac{1}{60}$	$\frac{7}{180}$	$1/12$
	$\frac{1}{3}$	$1/5$	$7/15$	

$E(X) = 0(1/3) + 1(1/5) + 2(7/15) = \frac{17}{15}$

$E(Y) = 0(1/4) + 1(2/3) + 2(1/12) = \frac{5}{6}$

$E(X)E(Y) = \frac{17}{\cancel{15}_{3}} \cdot \frac{\cancel{5}^{1}}{6} = \frac{17}{18}$

$E(XY) = 0(1/12 + 1/20 + 7/60 + 2/9 + \frac{1}{36}) + 1(2/15) + 2(1/60 + \frac{14}{45}) + 4(\frac{7}{180})$

$= \frac{17}{18}$

the values of E(X)E(Y) = E(XY) so they are independent

23. Prove that if $3n + 2$ is even, then n is even. (12 points)

if $3n+2$ is even then $3n+2 = 2k$ for some k

$\Rightarrow 3n = 2k-2 \Rightarrow 2(k-1) \therefore 3n$ is even.

Since 3 doesn't divide by 2, n must so n is even.

Q.E.D.

24. Find the $\mathcal{P}(A)$ if $A = \{m, n, o, p\}$. (6 points)

$\{\emptyset, \{m\}, \{n\}, \{o\}, \{p\}, \{m,n\}, \{m,o\}, \{m,p\}, \{n,o\}, \{n,p\}, \{o,p\},$
$\{m,n,o\}, \{m,n,p\}, \{n,o,p\}, \{m,n,o,p\}\}$

25. If $A_n = \left\{\frac{i}{n}\right\}_{i=1}^{n}$ for some integer, find $\bigcup_{n=1}^{4} A_n$. (8 points)

$A_1 = \{1\}$ $A_2 = \{\frac{1}{2}, 1\}$ $A_3 = \{1/3, 2/3, 1\}$ $A_4 = \{\frac{1}{4}, 1/2, 3/4, 1\}$, etc.

$\bigcup_{n=1}^{4} A_n = \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 2/3, 3/4, 1\}$

26. Prove that $\overline{A \cap B} = \bar{A} \cup \bar{B}$. You may not depend on a Venn diagram as your only 'proof'. Use elements of the set. (12 points)

let $x \in \overline{A \cap B}$. This means that x is not in A & B both.

therefore $x \in \bar{A}$ or $x \in \bar{B}$, but this implies $x \in \bar{A} \cup \bar{B}$.

So $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$

Suppose $x \in \bar{A} \cup \bar{B}$ then $x \in \bar{A}$ or $x \in \bar{B}$. if $x \in \bar{A}$, then it's not in $A \cap B$ and so it is in $\overline{A \cap B}$. if $x \in \bar{B}$, then x is not $A \cap B$ and so is in $\overline{A \cap B}$. therefore $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$.

but since $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$ & $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$, This is only possible if $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

Q.E.D.