Math 2366 Proof Set #3 Key

1. Show that $\binom{n}{k} \le \frac{n^k}{2^{k-1}}$

if $k=1$ then $\binom{n}{1} \le \frac{n^1}{2^0}$ both sides $= n$ so the base case is true.

Suppose it's true for some k that $\binom{n}{k} \le \frac{n^k}{2^{k-1}}$ and prove true for $\binom{n}{k+1} \le \frac{n^{k+1}}{2^k}$. if $\binom{n}{k} \le \frac{n^k}{2^{k-1}}$ Then $\frac{n!}{k!(n-k)!} \le \frac{n^k}{2^{k-1}}$

$$\binom{n}{k+1} = \frac{n!}{(k+1)!\,(n-k-1)!} = \frac{n!}{k!(k+1)(n-k-1)!}\frac{(n-k)}{(n-k)} = \frac{n!}{k!(n-k)!}\cdot\frac{n-k}{k+1} = \binom{n}{k}\cdot\frac{n-k}{k+1}$$

if $\binom{n}{k} \le \frac{n^k}{2^{k-1}}$ then $\binom{n}{k}\cdot\frac{n-k}{k+1} \le \frac{n^k}{2^{k-1}}\cdot\frac{n-k}{k+1} = \frac{n^{k+1}}{(2^{k-1})(k+1)} - \frac{n^k}{2^{k-1}}\left(\frac{k}{k+1}\right)$

the smallest value k can be is 1 so $k+1 \ge 2$ so

$\frac{n^{k+1}}{2^{k-1}(k+1)} \le \frac{n^{k+1}}{2^k}$ and since $\frac{n^{k+1}}{2^{k-1}(k+1)} - \frac{n^k}{2^{k-1}}\left(\frac{k}{k+1}\right) < \frac{n^{k+1}}{(k+1)2^{k-1}}$

since $\left(\frac{k}{k+1}\right) > 0$ and always < 1. Therefore we can conclude that

$$\binom{n}{k+1} = \binom{n}{k}\frac{n-k}{k+1} \le \frac{n^k}{2^{k-1}}\left(\frac{n-k}{k+1}\right) < \frac{n^{k+1}}{2^{k-1}(k+1)} < \frac{n^{k+1}}{2^k}$$

which was to be shown.

2. Show $\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \frac{1}{2}\binom{2n+2}{n+1} - \binom{2n}{n}$

using identity $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$ and $\binom{2n+2}{n+1} = \binom{2(n+1)}{n+1}$

the right side becomes $\frac{1}{2}\sum_{k=0}^{n+1}\binom{n+1}{k}^2 - \sum_{k=0}^{n}\binom{n}{k}^2$. Pascal's identity says $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ and that $\sum_{k=0}^{n+1}\binom{n+1}{k}^2 = \sum_{k=1}^{n}\binom{n+1}{k}^2 + \binom{n+1}{0} + \binom{n+1}{n+1}$

we obtain $\frac{1}{2}\sum_{k=1}^{n}\binom{n+1}{k}^2 + \frac{1}{2}(2) - \left[\sum_{k=1}^{n}\binom{n}{k}^2 + 1\right] = \frac{1}{2}\sum_{k=1}^{n}\left[\binom{n}{k-1} + \binom{n}{k}\right]^2 - \sum_{k=1}^{n}\binom{n}{k}^2$

then FOILING expression inside fis sum:

$$\frac{1}{2}\sum_{k=1}^{n}\left[\binom{n}{k-1} + \binom{n}{k}\right]^2 = \frac{1}{2}\sum_{k=1}^{n}\left[\binom{n}{k-1}^2 + 2\binom{n}{k-1}\binom{n}{k} + \binom{n}{k}^2\right] - \sum_{k=1}^{n}\binom{n}{k}^2$$

$$= \sum_{k=1}^{n}\left[\frac{1}{2}\binom{n}{k-1}^2 + \binom{n}{k-1}\binom{n}{k} + \frac{1}{2}\binom{n}{k}^2 - \binom{n}{k}^2\right] = \sum_{k=1}^{n}\left[\frac{1}{2}\binom{n}{k-1}^2 + \binom{n}{k-1}\binom{n}{k} - \frac{1}{2}\binom{n}{k}^2\right] = \sum_{k=1}^{n}\binom{n}{k-1}\binom{n}{k} + \frac{1}{2}\sum_{k=1}^{n}\binom{n}{k-1}^2 - \frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}^2 =$$

$$\sum_{k=1}^{n}\binom{n}{k-1}\binom{n}{k} + \frac{1}{2}\sum_{k=0}^{n-1}\binom{n}{k}^2 - \frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}^2 = \sum_{k=1}^{n}\binom{n}{k-1}\binom{n}{k} + \frac{1}{2}\binom{n}{0} + \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}^2$$

$$- \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}^2 - \frac{1}{2}\binom{n}{n} = \sum_{k=1}^{n}\binom{n}{k-1}\binom{n}{k} + \frac{1}{2}(1) + \frac{1}{2}\sum_{k=1}^{n-1}\left[\binom{n}{k}^2 - \binom{n}{k}^2\right] - \frac{1}{2}(1)$$

$$= \sum_{k=1}^{n}\binom{n}{k-1}\binom{n}{k}$$ which was to be shown.

3. for $1 \le k \le n$ $\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$.

the left side is equivalent to $\frac{(n-1)!}{(k-1)!\,(n-k+1)!} \cdot \frac{n!}{(k+1)!\,(n-k-1)!} \cdot \frac{(n+1)!}{k!\,(n-k)!}$

$= \frac{(n-1)!}{k!\,(n-k-1)!} \cdot \frac{n!}{(k-1)!\,(n-(k-1))!} \cdot \frac{(n+1)!}{(k+1)!\,(n-k)!} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$.

4. Let $p+q=1$ for any bernoulli random variable where p is the probability of the event and $q=1-p$ is the probability of the complement. then for n trials of the event $(p+q)^n$ represents all the possible outcomes, but $(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$

$= \binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \cdots \binom{n}{n} p^n q^0$ but since $p+q=1$

$(p+q)^n = 1$ as well so $\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = 1$ as it should for the sum of probabilities over a sample space.

5. for R to be symmetric, $(a,b) \in R$ implies (b,a) in R for all combinations of a & b. R^{-1} is defined to be $(x,y) \in R$ then (y,x) in R^{-1}! But since R is symmetric, this implies that (y,x) in R and by consequence so (x,y) also in R^{-1}. Since this applies to also possible coordinate pairs, $R = R^{-1}$.

6. Show $x \oplus y = x\bar{y} + \bar{x}y$

We will show this by truth tables.

x	y	$x \oplus y$	$\bar{y}$	$x\bar{y}$	$\bar{x}$	$\bar{x}y$	$x\bar{y} + \bar{x}y$
1	1	0	0	0	0	0	0
1	0	1	1	1	0	0	1
0	1	1	0	0	1	1	1
0	0	0	1	0	1	0	0
		*					*

Since the two starred columns are the same values for the same truth values of x and y, the statements are equivalent.

7. To show $\bar{\bar{x}} = x$ we can show this by supposing x is true. Thus $x=1$. Then $\bar{x} = 0$ and $\overline{(\bar{x})} = \overline{(0)} = 1$ Thus if $x=1$, $\bar{\bar{x}}=1$. Suppose x is false, thus $x=0$. Then $\bar{x} = 1$ and $\overline{(\bar{x})} = \overline{(1)} = 0$ thus if $x=0$, $\bar{\bar{x}}$ also $=0$. Therefore $\bar{\bar{x}} = x$.

(This can also be done by a very short truth table.)