Proof Set #1 Key Math 2366

1.

p	q	$p\leftrightarrow q$	$p\to q$	$q\to p$	$(p\to q)\wedge(q\to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

if $p\leftrightarrow q$ and $(p\to q)\wedge(q\to p)$ are logically equivalent then their truth values will be the same everywhere for the same inputs. The columns representing these expressions are identical and so they are logically equivalent.

2. if x is rational then it can be written as the ratio of two integers $\frac{p}{q}$ where $p,q\in\mathbb{Z}$ and $q\neq 0$. if $x=\frac{p}{q}$ then $\frac{1}{x}=\frac{1}{p/q}\Rightarrow\frac{1}{p/q}\cdot\frac{q}{q}=\frac{q}{p}$. which is also the ratio of two integers. However, note that p must also be non-zero! Assuming $p\neq 0$ also, then $1/x$ is rational.

3. Suppose that there is a rational root for the equation $r^3+r+1=0$. If r is rational then it can be expressed as $\frac{p}{q}$ for some set of integers. Let us further assume that the $\frac{p}{q}$ chosen is fully reduced and p & q have no common factors. Then we can rewrite the equation $r^3+r+1=0$ as $\left(\frac{p}{q}\right)^3+\left(\frac{p}{q}\right)+1=0$ multiplying both sides by q^3 we obtain the equation $p^3+pq^2+q^3=0$. Since zero is even and p and q must be even or odd, there are 4 cases we can check.

Suppose that p and q are both odd. Then p^3 is odd, so is

3 cont'd.

pq^2 and so is q^3. The sum of 3 odd numbers is odd and so odd number = even number is impossible.

The next two cases are where p and q are one even and one odd. Suppose p is even and q is odd. Then p^3 is even. pq^2 is even and q^3 is odd. The sum of two even numbers is even, and the sum of an even number and an odd number is odd, but this is supposed to be equal to 0 which is even. Again, this is a contradiction.

Without loss of generally, The same results occur when p is odd and q is even.

The last scenario is for p and q both even, but this violates our original assumption that the two numbers have no common factor.

Having exhausted all the possibilities for rational roots, and obtained a contradiction in each case, we must conclude that there is no rational root for the equation $r^3+r+1=0$.

4. For $|x|+|y| \geq |x+y|$ consider the following cases: 1) Suppose x and y have the same sign 2) suppose x and y will have opposite signs.

If x and y have the same sign (for instance positive), then you will be summing the magnitude of both numbers on each side and the equality symbol will hold. $|x|=x$, $|y|=y$ and $|x+y|=x+y$ and so $|x|+|y|=|x+y|$ since $x+y=x+y$

If both signs are negative then the absolute value will change

4 cont'd

the signs $|x| = -x$, $|y| = -y$ and $|x+y| = -(x+y) = -x-y$ and so again we'd have equality since $|x|+|y| = -x+(-y)$ $= -x-y$ and that is the same as $|x+y| = -x-y$.

In the second case, if the signs are opposite then, suppose the $x>y$. then x would be positive and y negative, so $|x| = x$ but $|y| = -y$ However $|x+y|$ would be smaller than $x-y$ since y is negative. Thus $|x|+|y| > |x+y|$. without loss of generality, we can say the same if $y>x$.

Therefore the triangle inequality $|x|+|y| \geq |x+y|$ holds for all x and y.

5. Let's consider the possible perfect squares and cubes less than 100:

perfect squares: $n=1 \Rightarrow n^2=1$; $n=2 \Rightarrow n^2=4$; $n=3 \Rightarrow n^2=9$;
$n=4 \Rightarrow n^2=16$; $n=5 \Rightarrow n^2=25$; $n=6 \Rightarrow n^2=36$; $n=7 \Rightarrow n^2=49$;
$n=8 \Rightarrow n^2=64$; $n=9 \Rightarrow n^2=81$.

perfect cubes: $n=1 \Rightarrow n^3=1$; $n=2 \Rightarrow n^3=8$; $n=3 \Rightarrow n^3=27$; $n=4 \Rightarrow n^3=64$.

That's it, so we can check the list exhaustively

if $n=1$ $n^2+n^3 = 1+1 = 2 \neq 100$
if $n=2$ $n^2+n^3 = 4+8 = 12 \neq 100$
if $n=3$ $n^2+n^3 = 9+27 = 36 \neq 100$
if $n=4$ $n^2+n^3 = 16+64 = 80 \neq 100$

all the remaining squares under 100 have cubes that are already greater than 100, so we don't need to check those. Since there are no other positive integers w/ squares or cubes less than 100, we can conclude that no integer can satisfy the equation $n^2+n^3=100$.

6. To show that $\sqrt[3]{2}$ is irrational, we will show it cannot be rational by proof by contradiction.
Suppose that $\sqrt[3]{2}$ is rational, i.e. that it can be written as $\frac{p}{q}$ for integers p and q, and $q \neq 0$. Further suppose that $\frac{p}{q}$ is fully reduced. $\sqrt[3]{2} = \frac{p}{q}$ then $2 = \frac{p^3}{q^3}$ and $2q^3 = p^3$. Since q^3 is an integer $2q^3$ must be even. That's only possible if p^3 is even and that means p must also be even. let $p = 2k$ where k is an integer. then $p^3 = (2k)^3 = 8k^3$ so we have $2q^3 = 8k^3$ and dividing by 2 obtain $q^3 = 4k^3$. But $4k^3$ is even and so this implies that q^3 must also be even and consequently, so must q be. But if q is even and p is even then they have a common factor of 2 and so are not fully reduced. This is a contradiction, so it must be the case that $\sqrt[3]{2}$ is irrational.

7. Let's suppose that $A = B$. and let $a_i \in A$, and $b_j \in B$. and let $a_n = b_n$ for all n. Then for each pair (a_i, b_j) in $A \times B$ there will be an identical pair (b_i, a_j) in $B \times A$. This accounts for all the elements of $A \times B$ and $B \times A$, so $A \times B = B \times A$.
For the other scenarios let's assume that at least one $b_i \neq a_i$. Then there will be a point (a_i, b_i) in $A \times B$ and no such point in $B \times A$. $B \times A$ will have the point (b_i, a_i) but $A \times B$ will not. This is exaserbated by the sizes of A and B are not the same. Therefore $A \times B \neq B \times A$ unless $A = B$.