2366 ~~Proof~~ Set #2 Key

1. to show that $\overline{A\cup B} = \bar{A}\cap\bar{B}$ we must show that $\overline{A\cup B} \subset \bar{A}\cap\bar{B}$ & that $\bar{A}\cap\bar{B} \subset \overline{A\cup B}$.

let $x \in \overline{A\cup B}$ Then x is not an element of either A or B. This means that x is in $\bar{A}$ and in $\bar{B}$, so $x \in \bar{A}\cap\bar{B}$. $\therefore$ $\overline{A\cup B} \subset \bar{A}\cap\bar{B}$

let $x \in \bar{A}\cap\bar{B}$, then x in both the complement of A, $\bar{A}$, and the complement of B, $\bar{B}$. Since x is neither in A nor B, x is not in $A\cup B$, so x is in $\overline{A\cup B}$. Therefore $\bar{A}\cap\bar{B} \subset \overline{A\cup B}$.

the mutual subset relation is only possible if $\overline{A\cup B} = \bar{A}\cap\bar{B}$.

2. to show that $A-B = A\cap\bar{B}$ we show $A-B \subset A\cap\bar{B}$, and $A\cap\bar{B} \subset A-B$.

Suppose $x \in A-B$, then x is in A, but not in B. Since x is not in B, it is in $\bar{B}$ and since x is both in A & $\bar{B}$, then $x \in A\cap\bar{B}$. $\therefore$ $A-B \subset A\cap\bar{B}$.

Suppose $x \in A\cap\bar{B}$. Then x is both in A and in $\bar{B}$. If x is in A, but not in B, it is in $A-B$. $\therefore$ $A\cap\bar{B} \subset A-B$.

$\therefore$ The mutual subset relation is only possible if $A-B = A\cap\bar{B}$.

3. Show that $|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$

We begin by considering the two-set case $|A\cup B| = |A| + |B| - |A\cap B|$.

the three set case follows with some set properties.

$|A\cup B\cup C| = |(A\cup B)\cup C| = |A\cup B| + |C| - |(A\cup B)\cap C| = |A| + |B| - |A\cap B| + |C| - |(A\cup B)\cap C|$. by the distributive property, this last set relation can be written as $|(A\cup B)\cap C| = |(A\cap C)\cup(B\cap C)|$. and reapplying the two-set relation, this becomes $|A\cap C| + |B\cap C| - |(A\cap C)\cap(B\cap C)|$ but this last expression is just $|A\cap B\cap C|$. Substituting gives us

3 cont'd

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - \left[|A \cap C| + |B \cap C| - |A \cap B \cap C|\right] =$

$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$

4. $\sum_{i=0}^{n} (2i+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3.}$ to prove, we use mathematical induction

the base case is $\sum_{i=0}^{0} 1^2 = 1 = \frac{(1)(1)(3)}{3} = 1$

to prove the inductive step, we assume it is true for k, i.e.

$\sum_{i=0}^{k} (2i+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$ and show it is true for k+1

$\sum_{i=0}^{k+1} (2i+1)^2 = \sum_{i=0}^{k} (2i+1)^2 + (2(k+1)+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2$

$(2k+3)\left[\frac{(k+1)(2k+1)}{3} + \frac{3(2k+3)}{3}\right] = \frac{2k+3}{3}\left[2k^2+k+2k+1+6k+9\right]$

$= \frac{2k+3}{3}\left[2k^2+9k+10\right] = \frac{2k+3}{3}(2k+5)(k+2)$

which is what we expected $\sum_{i=0}^{k+1} (2i+1)^2 = \frac{(k+1+1)(2(k+1)+1)(2(k+1)+3)}{3}$

$= \frac{(k+2)(2k+3)(2k+5)}{3.}$ Therefore, it is true for all $n \geq 0$.

5. $\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$ for $n > 1$ to prove, we use mathematical induction.

base case let $n=2$ $\sum_{i=1}^{2} \frac{1}{i^2} = 1 + \frac{1}{4} = \frac{5}{4} < 2 - \frac{1}{2} = \frac{3}{2}$ true.

if we assume this holds for all integers up to k, we must show the next case k+1 also follows.

$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$ for all k since

$\left[-\frac{1}{k} + \frac{1}{(k+1)^2} < -\frac{1}{k+1}\right] k(k+1)^2 \Rightarrow \cancel{-k^2} - 2k - 1 + k < \cancel{-k^2} - k \Rightarrow$

$-\cancel{k} - 1 < \cancel{-k} \Rightarrow -1 < 0$ which is true.

Therefore, $\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$ for all $n > 1$.

6. If n is an odd positive integer, we can write this as $n = 2k+1$ for $k \geq 0$. Then $n^2-1 = (2k+1)^2 - 1 = 4k^2+4k$.

Suppose $k=0$ then $n^2-1 = 0$ and 0 is divisible by 8.

for $k>0$ if k is even then $k = 2l$ $n^2-1 = 4(2l)^2+4(2l) = 16l^2+8l = 8(2l^2+l)$ and so n^2-1 is divisible by 8.

if k is odd, then $k = 2l+1$ & $n^2-1 = 4(2l+1)^2 + 4(2l+1) = 4(4l^2+4l+1) + 8l+4 = 16l^2+16l+4+8l+4 = 16l^2+24l+8 = 8(2l^2+3l+1)$ which is also divisible by 8. $\therefore$ n^2-1 is divisible by 8 for all $n \geq 0$ which are also odd.

7. The number of possible initials with three letters is 26^3, and the number of possible birthdates is 366 (we'll count leap day as "worst case" scenario. There are 6,432,816 possible combinations of initials and date of birth. Dividing 37 million by this figure we get 5.75175... This means that if we don't allow repeating initial-birthdate combinations to repeat until we've used up all possible cases, the "boxes" will already have 5 people in each "box", with each combination of initials & birthdates by the time we get to about 32,164,080 people. The 32,164,081st person will mean there is at least one combination w/ at least 6 people having that same combination of initials & birthdates.