Math 2415, Exam #3, Summer 2014 Name KEY

Instructions: Show all work. Justify answers as completely as possible. If you are asked to prove something, mere computation is not enough. You must explain your reasoning. Be sure to state your conclusion clearly. Incomplete work or justification will not receive full credit. Use exact answers unless specifically asked to round.

1. Find the eigenvalues and eigenfunctions for the differential equation $y'' + \lambda y = 0, y'(0) = 0$, $y'(L) = 0$. Be sure to check all three cases ($\lambda < 0, \lambda = 0, \lambda > 0$). Determine if the solution in each case exists, is trivial, and/or is unique. (20 points)

$r^2 + \lambda = 0$

$r = \pm\sqrt{-\lambda}$

if $\lambda = 0 \Rightarrow y'' = 0$

$y = At + B$

$y' = A \Rightarrow A = 0$

$y = B$

$y' = 0 \Rightarrow B$ can be anything

$y = B$

if $\lambda = -\mu^2$

$r = \pm\mu$

$y = c_1 e^{\mu t} + c_2 e^{-\mu t}$

$y' = c_1\mu e^{\mu t} - c_2\mu e^{-\mu t}$

$(c_1 - c_2)\mu = 0$

$c_1 = c_2$

$c_1\mu(e^{\mu L} - e^{-\mu L}) = 0$

$(e^{\mu L} = e^{-\mu L})e^{\mu L}$

$e^{2\mu L} = 1$

$2\mu L = 0 \Rightarrow \mu = 0$

See prev. case

no solution for $\mu > 0$

$c_1 = c_2 = 0$

trivial

if $\lambda = \mu^2$

$r = \pm\mu i$

$y = c_1 \cos\mu t + c_2 \sin\mu t$

$y' = -c_1\mu \sin\mu t + c_2\mu\cos\mu t$

$y'(0) = 0: c_2 = 0$

$y' = -c_1\mu \sin\mu L = 0$

$\mu L = n\pi$

$\mu = \frac{n\pi}{L}$

c_1 is not unique

$y = c_1 \cos\left(\frac{n\pi}{L}t\right)$

2. Give examples of two functions with a fixed period (and state the period for each), and two functions that are fundamentally non-periodic. (8 points)

periodic

$\cos 2t$ (period $= \pi$)

$\tan t$ (period $= \pi$)

nonperiodic

x^2, e^x

answers will vary

3. Find the Fourier series for the function $f(x) = \begin{cases} x, & -\pi \le x < 0 \\ 0, & 0 \le x < \pi \end{cases}$, $f(x+2\pi) = f(x)$. Be sure to simplify the coefficients as much as possible. (25 points)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx = \frac{1}{\pi}\cdot\frac{1}{2}x^2\Big|_{-\pi}^{0} = \frac{1}{2\pi}\left[0-\pi^2\right] = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} x\cos(nx)\,dx = \frac{1}{\pi}\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right|_{-\pi}^{0}$$

$$\frac{1}{\pi}\left[0 + \frac{1}{n^2} - \frac{-\pi}{n}(0) - \frac{1}{n^2}\cos(n\pi)\right] = \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right] =$$

$$\frac{1}{n^2\pi} - \frac{(-1)^n}{n^2\pi} = \frac{1-(-1)^n}{n^2\pi}$$ n even $\Rightarrow 0$, n odd $\Rightarrow 2$ $= \frac{2}{(2k+1)^2\pi}$ or $\frac{2}{(2k-1)^2\pi}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} x\sin(nx)\,dx = \frac{1}{\pi}\left[\frac{-x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right)\Big|_{-\pi}^{0}$$

$$\frac{1}{\pi}\left[0 + \frac{1}{n^2}(0) + \frac{-\pi}{n}\cos(n\pi) - \frac{1}{n^2}(0)\right] = \frac{1}{\cancel{\pi}}\left[\frac{-\cancel{\pi}}{n}(-1)^n\right] = \frac{(-1)^n}{n}$$

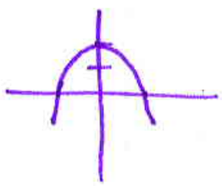
$$-\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{2}{(2n-1)^2\pi}\cos((2n-1)x) + \frac{(-1)^n}{n}\sin(nx)\right]$$

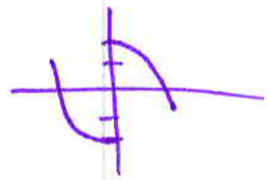
4. For each of the functions below, rewrite the function so that the resulting function is a) even, b) odd. Sketch the graph in each case. State the length of the period. (10 points each)

i. $f(x) = 2 - x^2,\ 0 \le x \le 2$

even

odd





$$f(x) = \begin{cases} 2 - x^2, & -2 \le x \le 2 \end{cases}$$

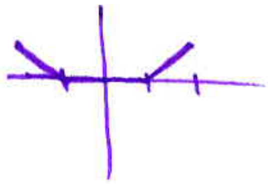
(already even)

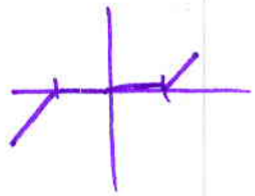
$$f(x) = \begin{cases} x^2 - 2 & -2 \le x < 0 \\ 2 - x^2 & 0 \le x < 2 \end{cases}$$

ii. $f(x) = \begin{cases} 0, & 0 \le x < 1 \\ x - 1, & 1 \le x < 2 \end{cases}$

even

odd





$$f(x) = \begin{cases} -x-1, & -2 \le x < -1 \\ 0, & -1 \le x < 1 \\ x-1, & 1 \le x < 2 \end{cases}$$

$$f(x) = \begin{cases} x+1 & -2 \le x < -1 \\ 0 & -1 \le x < 1 \\ x-1 & 1 \le x < 2 \end{cases}$$

5. Under what conditions does the Fourier series contain only sine functions? Under what conditions does it contain only cosine function? When must it contain both types of functions? Explain your reasoning. (9 points)

Since sine functions are all odd, odd functions are transformed into sine-only Fourier series (all $a_n=0$)

Since cosine functions are all even, even functions are transformed into cosine-only (give-or-take constant) Fourier series ($b_n=0$).

functions that are neither even nor odd must contain both sine and cosine functions in the Fourier transform.

6. Determine if the partial differential equations below can be solved with the method of separation of variables. (5 points each)

a. $u_{xx} + t^2 u_t = 0$

$u=XT \quad u_x = X'T \quad u_{xx}=X''T \quad u_t = XT'$

$\frac{X''\cancel{T}}{X} = \frac{-t^2\cancel{X}T'}{T} = \lambda$ Can be used. $u_{xt} = X'T'$

b. $u_{xx} + u_{xt} + u_t = 0$

$X''T + X'T' + XT' = 0$

$\frac{X''\cancel{T}}{\cancel{X}} = \frac{-T'(X'+X)}{T} \Rightarrow \frac{X''}{X'+X} = \frac{-T'}{T} = \lambda$ yes, it can be used.

c. $t^2 u_{xx} + x^3 u_{tt} = 0$

$$\frac{\cancel{t^2} X'' \cancel{T}}{X^3 X} = \frac{\cancel{x^3} \cancel{X} T''}{T t^2} = \lambda$$ yes it can be used

7. The solution to the heat equation is given by $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$ where $c_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$. Find the solution for the case when the rod is 50 cm long and both ends are maintained at 0°C for all $t > 0$. Suppose the initial temperature distribution is given by $u(x,0) = \begin{cases} 0, & 0 \le x < 10 \\ 40, & 10 \le x < 40 \\ 0, & 40 \le x \le 50 \end{cases}$. Assume that $\alpha^2 = 1$. (20 points)

$$c_n = \frac{1}{25}\int_{10}^{40} 40 \sin\left(\frac{n\pi x}{50}\right) dx = -\frac{40}{25} \cdot \frac{\overset{2}{\cancel{50}}}{n\pi} \cos\left(\frac{n\pi x}{50}\right)\Big|_{10}^{40}$$

$$\frac{-80}{n\pi}\left[\cos\left(\frac{4n\pi}{5}\right) - \cos\left(\frac{n\pi}{5}\right)\right]$$

$$u(x,t) = \frac{-80}{\pi} \sum \left[\frac{\cos\left(\frac{4n\pi}{5}\right) - \cos\left(\frac{n\pi}{5}\right)}{n}\right] e^{-\frac{n^2\pi^2 t}{2500}} \sin\left(\frac{n\pi x}{50}\right)$$

8. Consider a bar 30 cm long that is made of a material for which $\alpha^2 = 16$ and whose ends are insulated. Suppose that the initial temperature is zero except for the interval $5 < x < 10$, where the initial temperature is 25°C. Write the set of equations and initial conditions for the problem with proper notation. You do not need to solve. (6 points)

$$16 u_{xx} = u_t$$

$$u(x,0) = \begin{cases} 0 & 0 \le x \le 5 \\ 25 & 5 < x < 10 \\ 0 & 10 \le x \le 30 \end{cases}$$

$$\left.\begin{aligned} u_x(0,t) &= 0 \\ u_x(30,t) &= 0 \end{aligned}\right\} \text{insulated} \Rightarrow \text{no heat flow.}$$

9. Suppose that the solution to a Fourier series is given by $f(x) \approx \sum_{n=1}^{N} -\frac{2(-1)^n}{n\pi}\sin(n\pi x)$. Sketch the graph of the function using $N = 1, N = 2, N = 3, N = 5, N = 8$ (you will need 5 graphs, and you may use your calculator to obtain them). (20 points)

$N=1$

$$f(x) \approx \frac{-2(-1)}{\pi}\sin(\pi x) = \frac{2}{\pi}\sin(\pi x)$$

$N=2$

$$f(x) \approx \frac{2}{\pi}\sin(\pi x) - \frac{1}{\pi}\sin(2\pi x)$$

$N=3$

$$f(x) \approx \frac{2}{\pi}\sin(\pi x) - \frac{1}{\pi}\sin(2\pi x) + \frac{2}{3\pi}\sin(3\pi x)$$

$N=5$

$$f(x) \approx \frac{2}{\pi}\sin(\pi x) - \frac{1}{\pi}\sin(2\pi x) + \frac{2}{3\pi}\sin(3\pi x) - \frac{1}{2\pi}\sin(4\pi x) + \frac{2}{5\pi}\sin(5\pi x)$$

$N=8$

$$f(x) \approx \frac{2}{\pi}\sin(\pi x) - \frac{1}{\pi}\sin(2\pi x) + \frac{2}{3\pi}\sin(3\pi x) - \frac{1}{2\pi}\sin(4\pi x) + \frac{2}{5\pi}\sin(5\pi x) - \frac{1}{3\pi}\sin(6\pi x) + \frac{2}{7\pi}\sin(7\pi x) - \frac{1}{4\pi}\sin(8\pi x)$$