Math 2415, Final Exam, Summer 2014 Name KEY

**Instructions**: Show all work. Justify answers as completely as possible. If you are asked to prove something, mere computation is not enough. You must explain your reasoning. Be sure to state your conclusion clearly. Incomplete work or justification will not receive full credit. Use exact answers unless specifically asked to round.

1. Rewrite the differential equation $y'' + \frac{1}{2}y' + 2y = 3\sin t$ as a system of equations. (8 points)

$y = x_1$
$y' = x_2 = x_1'$
$x_2' = y''$

$x_2' = y'' = -\frac{1}{2}x_2 - 2x_1 + 3\sin t$
$x_1' = x_2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -2 & -1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3\sin t \end{bmatrix}$$

2. Solve the system of linear differential equations given by $\vec{x}' = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix}\vec{x}$. (12 points)

$(-2-\lambda)(4-\lambda)+5 = \lambda^2+2\lambda-4\lambda-8+5 = \lambda^2-2\lambda-3=0$

$(\lambda-3)(\lambda+1)=0 \qquad \lambda=3, \lambda=-1$

$\lambda = 3$

$\begin{bmatrix} -5 & 1 \\ -5 & 1 \end{bmatrix} \qquad \begin{matrix} -5x_1 = -x_2 \\ 5x_1 = x_2 \end{matrix} \qquad \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$\lambda = -1$

$\begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix} \qquad \begin{matrix} -x_1 = -x_2 \\ x_1 = x_2 \end{matrix} \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\vec{x} = c_1\begin{bmatrix} 1 \\ 5 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t}$$

3. For the system of linear differential equations $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \vec{x}$, find the eigenvalues and eigenvectors, and plot several sample trajectories along with any real eigenvectors. Does the origin attract, repel or is it a saddle point? (16 points)

$(1-\lambda)(-2-\lambda) - 4 = \lambda^2 + 2\lambda - \lambda - 2 - 4 = \lambda^2 + \lambda - 6 = 0$

$(\lambda+3)(\lambda-2) = 0$

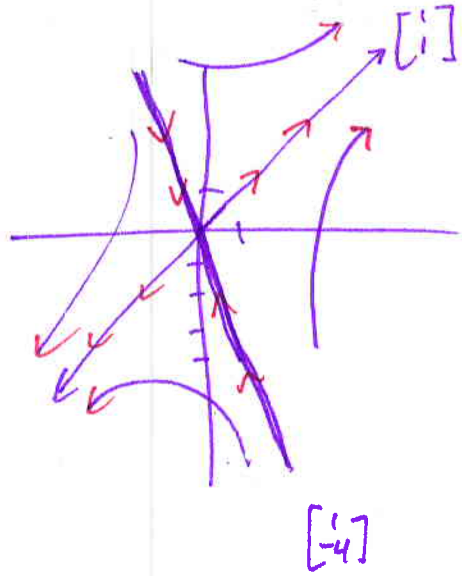
$\lambda = -3, 2$

$\lambda = -3$

$\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}$ $\quad 4x_1 = -x_2 \quad \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

$\lambda = 2$

$\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix}$ $\quad x_1 = x_2 \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$




origin: Saddle

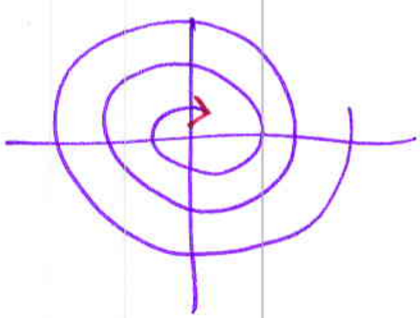
faster along trajectory $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$

So solutions will tend toward that direction

4. For the system of linear differential equations $\vec{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$, find the eigenvalues and eigenvectors, and plot several sample trajectories, along with any real eigenvectors. Does the origin attract, repel or is it a saddle point? (16 points)

$(-1-\lambda)(-1-\lambda) + 4 = \lambda^2 + 2\lambda + 1 + 4 = \lambda^2 + 2\lambda + 5 = 0$

$\lambda = \dfrac{-2 \pm \sqrt{4-20}}{2} = \dfrac{-2 \pm 4i}{2} = -1 \pm 2i$



$Re(\lambda) < 0$ So spirals into origin

5. Find the Fourier series for the function $f(x) = \begin{cases} x, & -\pi \le x < 0 \\ 0, & 0 \le x < \pi \end{cases}$, $f(x+2\pi) = f(x)$. Be sure to simplify the coefficients as much as possible. (25 points)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx = \frac{1}{\pi}\cdot\frac{1}{2}x^2\Big|_{-\pi}^{0} = \frac{1}{2\pi}\left[-\pi^2\right] = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} x\cos(nx)\,dx = \frac{1}{\pi}\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\Big|_{-\pi}^{0}\right. =$$

$$\frac{1}{\pi}\left[\frac{1}{n^2} - \frac{1}{n^2}\cos(n\pi)\right] = \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right] = \frac{1-(-1)^n}{n^2\pi}$$

$n$ even $\Rightarrow 0$
$n$ odd $\Rightarrow 2$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} x\sin(nx)\,dx = \frac{1}{\pi}\left[\frac{-x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\Big|_{-\pi}^{0}\right. =$$

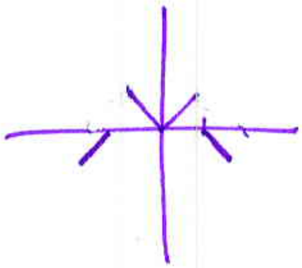
$$\frac{1}{\pi}\left[-\frac{\pi}{n}\cos(n\pi)\right] = \frac{(-1)^n}{n}$$

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{2}{(2n-1)^2\pi}\right]\cos((2n-1)x) + \frac{(-1)^n}{n}\sin(nx)$$

6. For the function below, rewrite the function so that the resulting function is a) even, b) odd. Sketch the graph in each case. State the length of the period. (12 points)

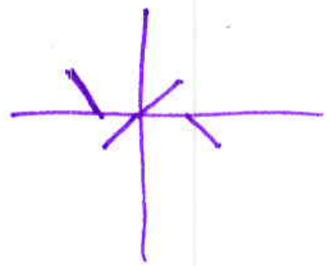
$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ -x + 1, & 1 \le x < 2 \end{cases}$$

even



$$f(x) = \begin{cases} x+1 & -2 \le x < -1 \\ -x & -1 \le x < 0 \\ x & 0 \le x < 1 \\ -x+1 & 1 \le x < 2 \end{cases}$$

odd



$$f(x) = \begin{cases} -x-1 & -2 \le x < -1 \\ x & -1 \le x < 1 \\ -x+1 & 1 \le x < 2 \end{cases}$$

7. Under what conditions does the Fourier series contain only sine functions? Under what conditions does it contain only cosine function? When must it contain both types of functions? Explain your reasoning. (12 points)

when the function is even. it will contain only cosine functions since cosines are also even.

when the function is odd, it will contain only sine functions since sine is odd.

When the function is neither even or odd, it contains both sines and cosines.

8. Determine if the partial differential equations below can be solved with the method of separation of variables. (8 points each)

a. $u_{xx} + t^2 u_t = 0$

$u = XT \quad u_x = X'T \quad u_{xx} = X''T \quad u_t = XT' \quad u_{xt} = X'T'$

$$\frac{X''\cancel{T}}{X} = \frac{-t^2\cancel{X}T'}{T} = \lambda$$

yes, it can be used

b. $u_{xx} + u_{xt} + u_t = 0$

$X''T + X'T' + XT' = 0$

$X''T = -T'(X'+X)$

$$\frac{X''}{X'+X} = \frac{-T'}{T} = \lambda$$

yes, it can be used.

9. Consider a bar 30 cm long that is made of a material for which $\alpha^2 = 16$ and whose ends are insulated. Suppose that the initial temperature is zero except for the interval $5 < x < 10$, where the initial temperature is 25°C. Write the set of equations and initial conditions for the problem with proper notation. You do not need to solve. (8 points)

$16 u_{xx} = u_t$

$$u(x,0) = \begin{cases} 0 & 0 \le x \le 5 \\ 25 & 5 < x < 10 \\ 0 & 10 \le x \le 30 \end{cases}$$

$u_x(0,t) = 0$

$u_x(30,t) = 0$

10. Solve the second order ordinary differential equations with constant coefficients. (10 points each)

a. $y'' + y' - 6y = 0$

$r^2 + r - 6 = 0$

$(r+3)(r-2) = 0$

$r = -3, r = 2$

$y = c_1 e^{-3t} + c_2 e^{2t}$

b. $4y'' + 12y' + 9y = 0$

$4r^2 + 12r + 9 = 0$

$(2r+3)^2 = 0$

$r = -3/2$

$y = c_1 e^{-3/2 t} + c_2 t e^{-3/2 t}$

c. $y'' + 4y' + 5y = 0$

$r^2 + 4r + 5 = 0$

$r = \frac{-4 \pm \sqrt{16-20}}{2}$

$= \frac{-4 \pm 2i}{2} = -2 \pm i$

$y = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$

11. Determine if the solutions $y_1 = t, y_2 = \sin t, y_3 = \cos t$ form a fundamental set by finding the value of the Wronskian. (10 points)

$$W = \begin{bmatrix} t & \sin t & \cos t \\ 1 & \cos t & -\sin t \\ 0 & -\sin t & -\cos t \end{bmatrix} = t(-\cos^2 t - \sin^2 t) - \sin t(-\cos t) + \cos(t)(-\sin t) =$$

$$-t + \sin t \cos t - \sin t \cos t = -t$$

$\neq 0$ for all $t \neq 0$

12. Suppose that the solutions to a second order differential equation are $y_1(t) = e^t, y_2(t) = e^{-2t}$. If the forcing term on the nonhomogeneous ODE is $F(t) = t^2 \sin t + e^{-2t} + 4\cosh t$, state your initial Ansatz for the method of undetermined coefficients (you do not need to solve for any of the coefficients, just state where you would start). (9 points)

$$(At^2 + Bt + C)\sin t + (Dt^2 + Et + F)\cos t + Gte^{-2t} + Hte^t + Ie^{-t}$$

$\cosh t = \frac{1}{2}(e^{-t} + e^t)$

13. Give an example of three functions that would need to be solved by the method of variation of parameters and cannot be solved by the method of undetermined coefficients. (9 points)

answers will vary

$x^n$ where $n$ is not a positive integer

trig functions other than sine & cosine

logs

14. Describe when a resonance occurs in a forced spring problem. (8 points)

resonance occurs when the forcing function has the same frequency as the underlying system

15. Describe when a transient solution exists in a spring problem. (8 points)

a transient solution exists when the system is damped and terms from the unforced system decrease to zero over time

16. How do the solutions of a spring system differ if the system is a) undamped, b) underdamped, c) critically damped, d) overdamped? (8 points)

undamped oscillates forever w/ only sine & cosine terms

underdamped has decaying oscillation w/ exponentials multiplied by trig functions

critically damped has two repeated roots $e^{rt}$ & $te^{rt}$

overdamped has two real exponentials that are different

17. Suppose that a mass of 10 kg stretches a spring 8 cm. Suppose that the mass is attached to a viscous damper with a damping constant of 40 Ns/m. If the mass is pulled down an additional 6 centimeters and then released with an upward velocity of 1 cm/s, find the differential equation and initial conditions to be used to solve for the position of the system. (You do not need to solve the equation, just set it up.) (10 points)

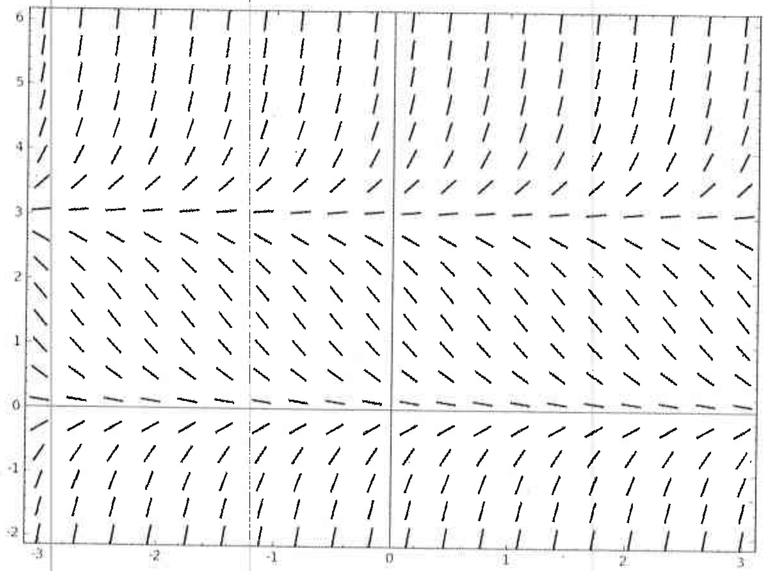
$m=10$ $\quad k=\frac{10 \cdot 9.8}{.08} = 1225 = \frac{245}{2}$ $\quad \gamma = 40$

$10y'' + 40y' + 1225y = 0$

$y(0) = -.06$

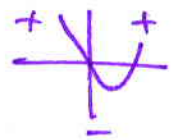
$y'(0) = .01$

18. Assuming the equilibrium solutions are integers, use the graph below to sketch the phase portrait of the differential equation that produced the slope field shown here, and write the differential equation that produced it. (12 points)




$y=3$ unstable

$y=0$ stable




$$\frac{dy}{dt} = y(y-3)$$

19. A tank has pure water flowing into it at 20 L/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 20 L/min. Initially, the tank contains 10 kg of salt in 1000 L of water. Find an equation to model the amount of salt in the tank at any time t. How much salt will there be in the tank after 30 minutes? (12 points)

$$\frac{dA}{dt} = \text{Rate in} - \text{Rate out} \qquad A(0) = 10$$

$$\frac{20L}{min} \cdot \frac{0L}{gal} - \frac{20L}{m} \cdot \frac{A}{1000}$$

$$\frac{dA}{dt} = -\frac{1}{50}A \Rightarrow \int \frac{dA}{A} = \int -\frac{1}{50}\,dt$$

$$\ln A = -\frac{1}{50}t + C$$

$$A = A_0 e^{-1/50 t} \qquad A_0 = 10$$

$$A(t) = 10e^{-1/50 t}$$

$$A(30) = 10e^{-\frac{30}{50}} \approx 5.488\text{ kg}$$

20. Solve the differential equation $y' = \frac{\sinh(x)}{3+4y}$, $y(0) = 1$. (12 points)

$$\int \sinh x \, dx = \int (3+4y)\,dy$$

$$\cosh x + C = 3y + \frac{4}{2}y^2$$

$$\cos(0) + C = 3(1) + 2(1)^2$$

$$1 + C = 5$$

$$C = 4$$

$$\cosh x + 4 = 2y^2 + 3y$$

21. Use the method of integrating factors to solve the differential equation $y' + \left(\frac{2}{t}\right)y = \frac{\sin t}{t^2}$, $y(\pi) = 0$. (12 points)

$$\mu = e^{\int 2/t \, dt} = e^{2\ln t} = t^2$$

$$t^2 y' + 2ty = \sin t$$

$$\int (t^2 y)' = \int \sin t$$

$$t^2 y = -\cos t + C$$

$$y = -\frac{\cos t}{t^2} + \frac{C}{t^2}$$

$$0 = \frac{-(-1)}{\pi^2} + \frac{C}{\pi^2} \Rightarrow C = -1$$

$$y = \frac{-\cos t - 1}{t^2}$$

22. Classify the following differential equations as a) ordinary or partial, b) order, c) linear or nonlinear. (5 points each)

a. $t^2\frac{d^3y}{dt^2} + t\frac{dy}{dt} + 2y = \tanh t$

linear, 3rd order, ordinary

b. $u_{xxy} + u_{xy} = e^u$

3rd order, nonlinear, partial