2415 Homework #4 Key

1a. $2y'' + 3y' + y = t^2 + 3\sin t \qquad y(0) = 0,\ y'(0) = 1$

$2r^2 + 3r + 1 = 0$

$(2r+1)(r+1) = 0 \Rightarrow r = -\frac{1}{2},\ r = -1 \qquad y_g(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{-t}$

$y_p(t) = At^2 + Bt + C + D\sin t + E\cos t$

$y_p'(t) = 2At + B + D\cos t - E\sin t$

$y_p''(t) = 2A + -D\sin t - E\cos t$

$2(2A - D\sin t - E\cos t) + 3(2At + B + D\cos t - E\sin t) + At^2 + Bt + C + D\sin t + E\cos t$

$4A - 2D\sin t - 2E\cos t + 6At + 3B + 3D\cos t - 3E\sin t + At^2 + Bt + C + D\sin t + E\cos t$

$At^2 = t^2 \Rightarrow A = 1$

$6At + Bt = 0 \Rightarrow 6 + B = 0 \Rightarrow B = -6$

$4A + 3B + C = 0 \Rightarrow 4 - 18 + C = 0 \Rightarrow C = +14$

$-2D\sin t - 3E\sin t + D\sin t = 3\sin t \Rightarrow -D - 3E = 3$

$-2E\cos t + 3D\cos t + E\cos t = 0 \Rightarrow \quad 3D - E = 0 \Rightarrow E = 3D$

$-D - 3(3D) = 3 \Rightarrow -10D = 3 \Rightarrow D = -\frac{3}{10}$

$E = -\frac{9}{10}$

$y_p(t) = t^2 - 6t + 14 - \frac{3}{10}\sin t - \frac{9}{10}\cos t$

$y(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{-t} + t^2 - 6t + 14 - \frac{3}{10}\sin t - \frac{9}{10}\cos t$

$y(0) = 0 = c_1 + c_2 + 14 - \frac{9}{10} \Rightarrow c_1 + c_2 = -13.1$

$y'(t) = -\frac{1}{2}c_1 e^{-\frac{1}{2}t} - c_2 e^{-t} + 2t - 6 - \frac{3}{10}\cos t + \frac{9}{10}\sin t$

$y'(0) = 1 = -\frac{1}{2}c_1 - c_2 - 6 - \frac{3}{10} \Rightarrow -\frac{1}{2}c_1 - c_2 = 7.3$

$c_1 + c_2 = -13.1$

$-\frac{1}{2}c_1 - c_2 = 7.3$

$\frac{1}{2}c_1 = -5.8$

$c_1 = -11.6$

$c_2 = -1.5$

$y(t) = -11.6e^{-\frac{1}{2}t} - 1.5e^{-t} + t^2 - 6t + 14 - \frac{3}{10}\sin t - \frac{9}{10}\cos t$

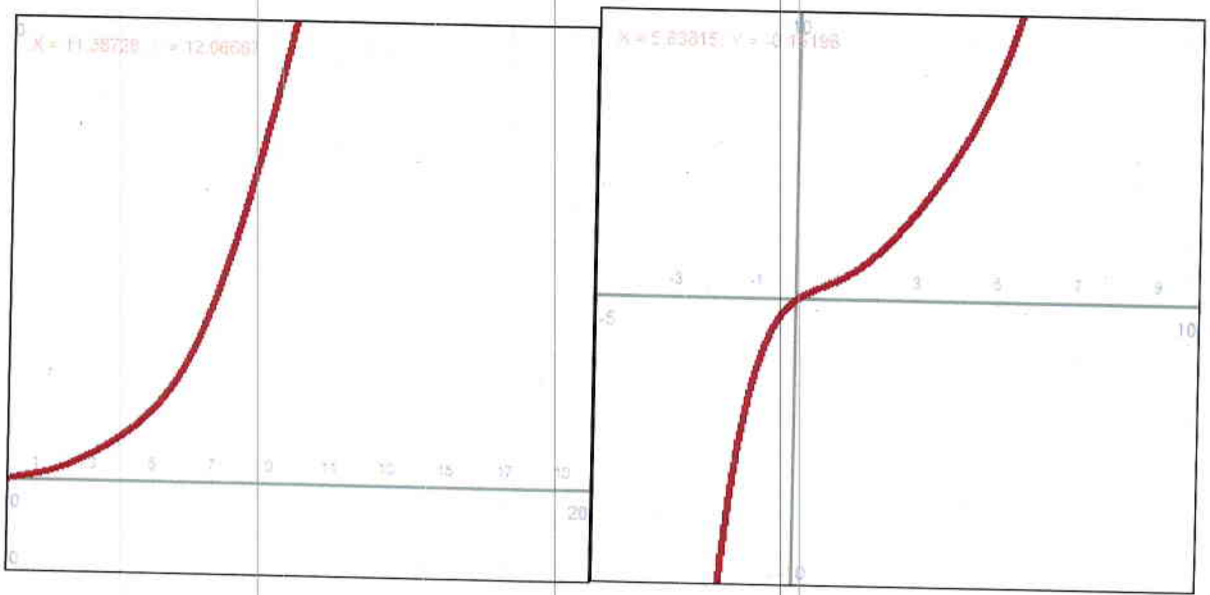
See attached graph

b. $y'' + y' + 4y = 2\sinh t \quad y(0) = 1,\ y'(0) = 0$

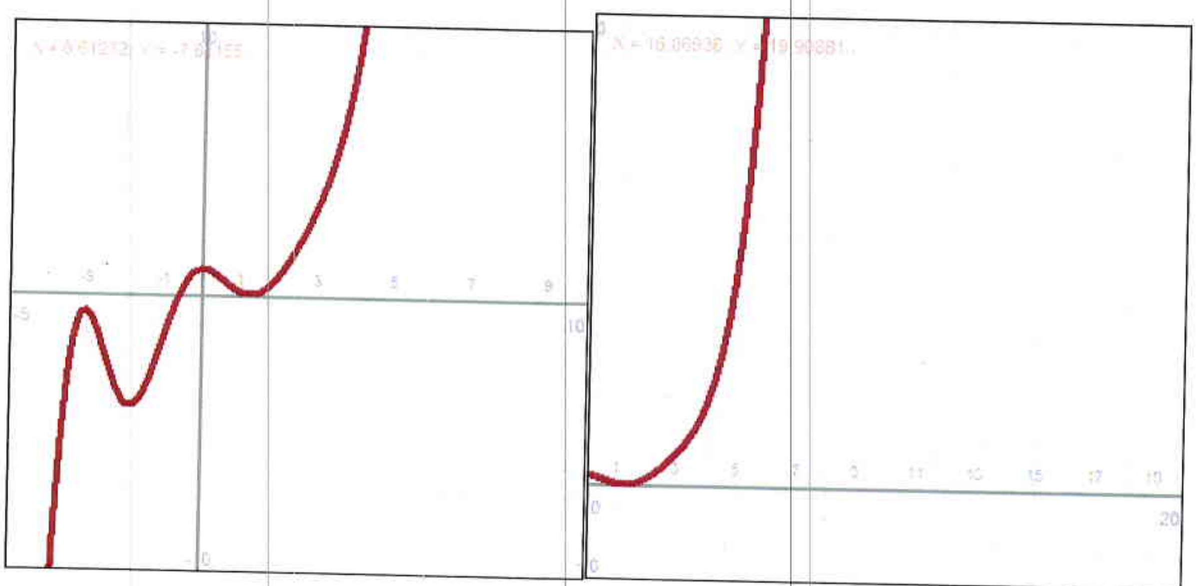
$r^2 + r + 4 = 0 \qquad \frac{-1 \pm \sqrt{1-16}}{2} = -\frac{1}{2} \pm \frac{\sqrt{15}}{2}i = r$

$y_g(t) = c_1 e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{15}}{2}t\right) + c_2 e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{15}}{2}t\right)$

1a.



1b.



1b cont'd

$Y_p(t) = A\sinh t + B\cosh t$

$Y_p'(t) = A\cosh t + B\sinh t$

$Y_p''(t) = A\sinh t + B\cosh t$

$A\sinh t + B\cosh t + A\cosh t + B\sinh t +$

$4A\sinh t + 4B\cosh t = 2\sinh t$

$5A + B = 2$

$A + 5B = 0 \Rightarrow A = -5B$

$5(-5B) + B = 2 \Rightarrow -24B = 2 \Rightarrow B = \frac{-1}{12}$

$\Rightarrow A = \frac{5}{12}$

$y(t) = c_1 e^{-1/2 t}\cos\left(\frac{\sqrt{15}}{2}t\right) + c_2 e^{-1/2 t}\sin\left(\frac{\sqrt{15}}{2}t\right) + \frac{5}{12}\sinh t - \frac{1}{12}\cosh t$

$y(0) = c_1 + -\frac{1}{12} = 1 \quad c_1 = \frac{13}{12}$

$y'(t) = -\frac{1}{2}c_1 e^{-1/2 t}\cos\left(\frac{\sqrt{15}}{2}t\right) - \frac{\sqrt{15}}{2}c_1 e^{-1/2 t}\sin\left(\frac{\sqrt{15}}{2}t\right) - \frac{1}{2}c_2 e^{-1/2 t}\sin\left(\frac{\sqrt{15}}{2}t\right) + \frac{\sqrt{15}}{2}c_2 e^{-1/2 t}\cos\left(\frac{\sqrt{15}}{2}t\right)$
$+ \frac{5}{12}\cosh t - \frac{1}{12}\sinh t$

$y'(0) = 0 = -\frac{1}{2}c_1 + \frac{\sqrt{15}}{2}c_2 + \frac{5}{12} \Rightarrow -\frac{13}{24} + \frac{5}{12} = -\frac{\sqrt{15}}{2}c_2 \Rightarrow c_2 = \frac{1}{4\sqrt{15}}$

$y(t) = \frac{13}{12}e^{-1/2 t}\cos\left(\frac{\sqrt{15}}{2}t\right) + \frac{1}{4\sqrt{15}}e^{-1/2 t}\sin\left(\frac{\sqrt{15}}{2}t\right) + \frac{5}{12}\sinh t - \frac{1}{12}\cosh t$

See attached graph

2.a. $r^2 + 2r + 2 = 0 \quad \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i \Rightarrow y_1 = e^{-t}\cos t \quad y_2 = e^{-t}\sin t$

$Y_p(t) = (At^3 + Bt^2 + Ct)(e^{-t}\sin t) + (Et^3 + Ft^2 + Gt)e^{-t}\cos t$

b. $y'' + 4y = t^2\sin 2t + (6t+7)\cos 2t$

$r^2 + 4 = 0 \Rightarrow r = \pm 2i \quad y_1 = \sin 2t \quad y_2 = \cos 2t$

$Y_p(t) = (At^3 + Bt^2 + Ct)\sin t + (Dt^3 + Et^2 + Gt)\cos t$

3.a. $y'' + y = \tan t \quad r^2 + 1 = 0 \Rightarrow r = \pm i \quad y_1 = \sin t \quad y_2 = \cos t$

$W = \begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix} = -1$

$y(t) = -\sin t \int \frac{\cos t \tan t}{-1}\,dt + \cos t \int \frac{\sin t \tan t}{-1}\,dt = +\sin t \int \cos t \frac{\sin t}{\cos t}\,dt + -$

$\cos t \int \frac{\sin t \sin t}{\cos t}\,dt = \sin t \int \sin t\,dt - \cos t \int \frac{1-\cos^2 t}{\cos t}\,dt = \sin t(-\cos t) - \cos t \int \sec t - \cos t\,dt$

$= -\sin t\cos t - \cos t \ln|\sec t + \tan t| + \sin t\cos t = -\cos t \ln|\sec t + \tan t|$

3a contd

$$y(t) = c_1 \sin t + c_2 \cos t - \cos t \ln|\sec t + \tan t|$$

3b. $y'' - 2y' + y = \frac{e^t}{1+t^2} \Rightarrow r^2 - 2r + 1 = 0 \quad (r-1)^2 = 0 \quad y_1 = e^t \quad y_2 = te^t$

$$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = e^{2t}$$

$$y(t) = -e^t \int \frac{te^t \cdot e^t}{e^{2t}(1+t^2)} dt + te^t \int \frac{e^t \cdot e^t}{e^{2t}(1+t^2)} dt = -e^t \int \frac{t}{1+t^2} dt + te^t \int \frac{1}{1+t^2} dt$$

$$= -\frac{1}{2} e^t \ln|1+t^2| + te^t \arctan t$$

$$y(t) = c_1 e^t + c_2 te^t - \frac{1}{2} e^t \ln|1+t^2| + te^t \arctan t$$

3c. $ty'' - (1+t)y' + y = t^2 e^{2t} \quad y_1 = 1+t \quad y_2 = e^t$

$$W = \begin{vmatrix} 1+t & e^t \\ 1 & e^t \end{vmatrix} = (1+t)e^t - e^t = e^t + te^t - e^t = te^t$$

$$y(t) = -(1+t) \int \frac{e^t \cdot t^2 e^{2t}}{te^t} dt + e^t \int \frac{(1+t)t^2 e^{2t}}{te^t} dt = -(1+t) \int te^{2t} dt + e^t \int e^t (1+t) dt$$

$$= -(1+t) t \cdot \frac{1}{2} e^{2t} + (1+t) \frac{1}{4} e^{2t} + e^t (1+t) e^t - e^t e^t$$

$$-\frac{1}{2} te^{2t} - \frac{1}{2} t^2 e^{2t} + \frac{1}{4} e^{2t} + \frac{1}{4} te^{2t} + \cancel{e^{2t}} + te^{2t} - \cancel{e^{2t}} = -\frac{1}{2} t^2 e^{2t} + \frac{3}{4} te^{2t} + \frac{1}{4} e^{2t}$$

$$y(t) = c_1 (1+t) + c_2 e^t - \frac{1}{2} t^2 e^{2t} + \frac{3}{4} te^{2t} + \frac{1}{4} e^{2t}$$

d. $x^2 y'' - 3xy' + 4y = x^2 \ln x \quad y_1 = x^2 \quad y_2 = x^2 \ln x$

$$W = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + 2x \end{vmatrix} = 2x^3 \ln x + 2x^3 - 2x^3 \ln x = 2x^3$$

$$y(t) = -x^2 \int \frac{x^2 \ln x \cdot x^2 \ln x}{2x^3} dx + x^2 \ln x \int \frac{x^2 \cdot x^2 \ln x}{2x^3} dx = -\frac{x^2}{2} \int x \ln^2 x \, dx + \frac{x^2 \ln x}{2} \int x \ln x \, dx$$

$u = \ln^2 x \quad v = x$
$du = 2 \ln x \cdot \frac{1}{x} \quad dv = \frac{1}{2} x^2$

$$x^2 \ln^2 x - \int x \ln x \, dx = \frac{1}{2} x^2 \ln^2 x - \frac{x^2}{2} \ln x + \int \frac{x}{2} dx = \frac{1}{2} x^2 \ln^2 x - \frac{x^2}{2} \ln x + \frac{x^2}{4}$$

$u = \ln x \quad dv = 2x$
$du = \frac{1}{x} \quad v = \frac{x^2}{2}$

$u = \ln x \quad dv = x$
$du = \frac{1}{x} \quad v = \frac{x^2}{2}$

$$\frac{1}{2} x^2 \ln x - \frac{x}{2} \qquad y(t) = -\cancel{\frac{x^4 \ln^2 x}{4}} + \frac{x^4}{4} \ln x - \frac{x^4}{8} + \cancel{\frac{x^4 \ln^2 x}{4}} - \frac{x^3 \ln x}{4}$$

$$y(t) = c_1 x^2 + c_2 x^2 \ln x + \frac{1}{4} x^4 \ln x - \frac{1}{8} x^4 - \frac{1}{4} x^3 \ln x$$

4. Undetermined coefficients (answers will vary)

e^t, te^t, $\sin t$, $t^2\cos t$, $\cosh t$, $t^3\sinh t$, $e^t\sin t$, etc.

variation of parameters (answers will vary)

$\tan t$, $\sec t$, $\ln t$, $x\ln x$, $\frac{1}{1+t^2}$, etc.

5. a. $R=\sqrt{3^2+4^2}=5$ $\quad \omega=2$ $\quad \delta=\tan^{-1}\left(\frac{4}{3}\right)\approx .9273$ radians or about $53.1°$

b. $R=\sqrt{2^2+3^2}=\sqrt{13}$ $\quad \omega=\pi$ $\quad \delta=\tan^{-1}\left(\frac{-3}{-2}\right)+\pi\approx 4.1244$ radians or about $236.3°$

(double negatives treat like pt in QIII)

6 $\quad .2y''+Ry'+\frac{1}{.8\times10^{-6}}y=0 \Rightarrow .2y''+Ry'+1{,}250{,}000y=0$

$$\frac{-R\pm\sqrt{R^2-4(.2)(1{,}250{,}000)}}{2(.2)}$$

$R^2-4(.2)(1{,}250{,}000)=0$

$R^2=10^6 \Rightarrow R=1000$

7. $.2y''+300y'+10^5y=0$ $\quad y(0)=10^{-6}$ $\quad y'(0)=0$

$.2r^2+300r+10^5=0$ $\quad r=\frac{-300\pm\sqrt{300^2-4(.2)10^5}}{.4}=\frac{-750\pm\sqrt{10000}}{.4}$

$=-750\pm250$ $\quad r=-500,\ r=-1000$

$y(t)=c_1e^{-500t}+c_2e^{-1000t}$ $\quad y(0)=10^{-6}=c_1+c_2$

$y'(t)=-500c_1e^{-500t}-1000c_2e^{-1000t}$ $\quad y'(0)=0=-500c_1-1000c_2$

$10^{-6}\times10^3=10^{-3} \Rightarrow 10^{-3}=1000c_1+1000c_2$

$0=-500c_1-1000c_2$

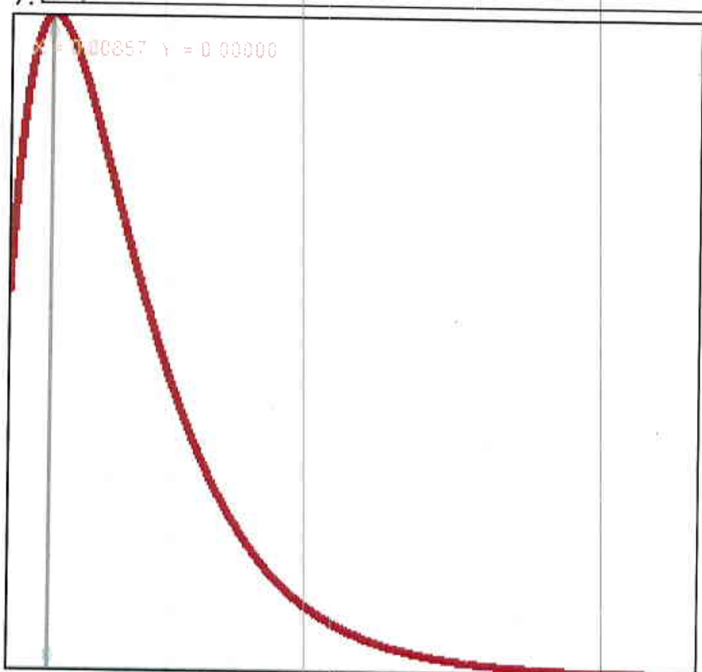
$10^{-3}=500c_1$ $\quad c_1=2\times10^{-6}$

$c_2=-1\times10^{-6}$

$y(t)=2\times10^{-6}e^{-500t}-10^{-6}e^{-1000t}$

See attached graph

7.



Scale is [0,10^{-6}] range in Y. t range is about [-0.001,0.015]