Math 2415 Homework #8

1a. $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 3 & 1 & 1 & 1 \\ -1 & 1 & 2 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 7/3 \\ 0 & 0 & 1 & -1/3 \end{array}\right] \quad \vec{x} = \frac{1}{3}\begin{bmatrix} -1 \\ 7 \\ -1 \end{bmatrix}$

b. $\left[\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ -2 & -4 & 2 & 4 \\ 2 & 4 & -2 & -4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_2 = \text{free} \\ x_3 = \text{free} \end{array}$

$$\begin{aligned} x_1 &= -2x_2 + x_3 - 2 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned} \Rightarrow \vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

2a. independent

b. dependent

c. dependent

d. dependent

e. $\begin{bmatrix} 2\sin t & \sin t \\ \sin t & 2\sin t \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \sin t \Rightarrow$ independent

3a. $\begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} = (-2-\lambda)^2 - 1 = \lambda^2 + 4\lambda + 4 - 1 = \lambda^2 + 4\lambda + 3 = 0$

$(\lambda+3)(\lambda+1) = 0 \qquad \lambda = -1, -3$

$\lambda = -1 \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \quad \begin{array}{l} x_1 = x_2 \\ x_2 = x_2 \end{array} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -3 \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \begin{array}{l} x_1 = -x_2 \\ x_2 = x_2 \end{array} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

b. $\begin{pmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{pmatrix} = (3-\lambda)\left[(4-\lambda)(-1-\lambda)+4\right] - 2\left[1(-1-\lambda)+2\right] + 2\left[-4+2(4-\lambda)\right]$

$(3-\lambda)\left[\cancel{-4} - 4\lambda + \lambda + \lambda^2 \cancel{+4}\right] - 2\left[-1-\lambda+2\right] + 2\left[-4+8-2\lambda\right]$

$(3-\lambda)(\lambda^2 - 3\lambda) + 2\lambda - 2 + 8 - 4\lambda$

$3\lambda^2 - \lambda^3 - 9\lambda + 3\lambda^2 - 2\lambda + 6 = -\lambda^3 + 6\lambda^2 - 11\lambda + 6$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\lambda = 1, \ \lambda = 2, \ \lambda = 3$

$\lambda=1$ $\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x_1 = -x_3,\ x_2 = 0,\ x_3 = x_3 \Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda=2$ $\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -4 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x_1 = -2x_2,\ x_2 = x_2,\ x_3 = 0 \quad \vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$\lambda=3$ $\begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -4 & -4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x_1 = 0,\ x_2 = -x_3,\ x_3 = x_3 \quad \vec{x} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

4. Hermitian matrices are symmetric $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$(1-\lambda)(1-\lambda) - 4 = 0$

$(1-\lambda)^2 = 4 \qquad 1-\lambda = \pm 2$

$\lambda - 1 = \pm 2 \qquad \lambda = 1 \pm 2 \Rightarrow \lambda = 3, 1$ both are real.

5. $W = \begin{pmatrix} e^t & t^2 \\ e^t & 2t \end{pmatrix} = 2te^t - t^2e^t = (2t - t^2)e^t = t(2-t)e^t$

linearly independent on $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$

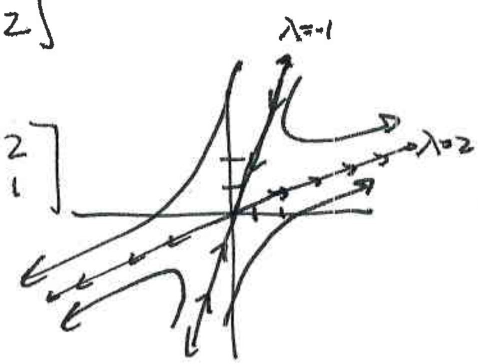
6. a. $\vec{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\vec{x} \qquad \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} \Rightarrow (3-\lambda)(-2-\lambda) + 4 = 0$

$-6 - 3\lambda + 2\lambda + \lambda^2 + 4 = 0$

$\lambda^2 - \lambda - 2 = 0 \Rightarrow (\lambda-2)(\lambda+1) = 0 \Rightarrow \lambda = 2, \lambda = -1$

$\lambda=-1$ $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$ $\quad 2x_1 = x_2,\ x_2 = x_2 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} x_2 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda=2$ $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$ $\quad x_1 = 2x_2,\ x_2 = x_2 \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_2 \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



as $t \to \infty$ $\quad x \to c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$

$x = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$

6b. $\vec{x}' = \begin{pmatrix}1 & 1\\ 4 & -2\end{pmatrix}\vec{x}$ $\quad \begin{pmatrix}1-\lambda & 1\\ 4 & -2-\lambda\end{pmatrix} \Rightarrow (1-\lambda)(-2-\lambda) - 4 = 0$

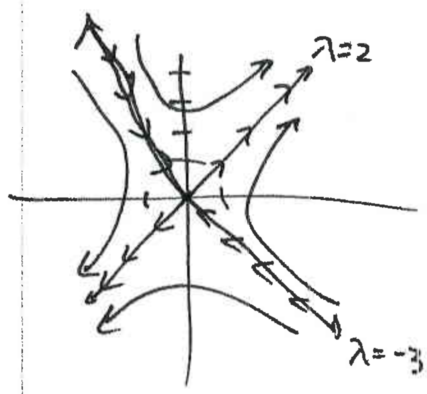
$-2-\lambda+2\lambda+\lambda^2-4=0 \Rightarrow \lambda^2+\lambda-6=0$

$(\lambda+3)(\lambda-2)=0 \quad \lambda=-3,\ \lambda=2$

$\lambda=-3$ $\begin{pmatrix}4 & 1\\ 4 & 1\end{pmatrix}$ $\quad \begin{matrix}4x_1=-x_2\\ x_2=x_2\end{matrix}$ $\quad \begin{bmatrix}-1\\4\end{bmatrix}=\vec{v}_1$ $\quad \lambda=2$ $\begin{pmatrix}-1 & 1\\ 4 & -4\end{pmatrix}$ $\quad \begin{matrix}x_1=x_2\\ x_2=x_2\end{matrix}$ $\quad \vec{v}_2=\begin{bmatrix}1\\1\end{bmatrix}$

as $t\to\infty,\ \vec{x} \to c_2\begin{pmatrix}1\\1\end{pmatrix}e^{2t}$

$x = c_1\begin{pmatrix}-1\\4\end{pmatrix}e^{-3t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{2t}$



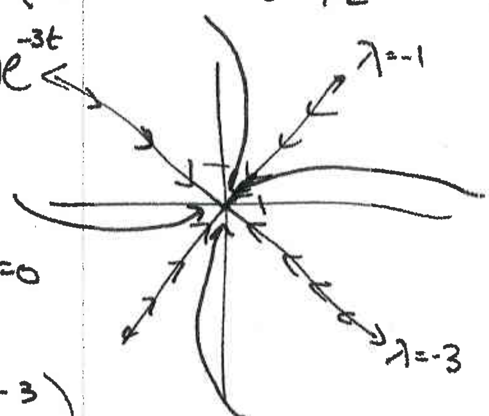
c. $\vec{x}' = \begin{pmatrix}-2 & 1\\ 1 & -2\end{pmatrix}\vec{x}$ $\quad \begin{pmatrix}-2-\lambda & 1\\ 1 & -2-\lambda\end{pmatrix} \Rightarrow$

$(-2-\lambda)^2 - 1 = 0 \quad (-2-\lambda)^2 = 1 \quad 2+\lambda=\pm 1$

$\lambda = -2\pm 1 \quad \lambda=-3,\ -1$

$\lambda=-1$ $\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}$ $\quad \begin{matrix}x_1=x_2\\ x_2=x_2\end{matrix}$ $\quad \begin{bmatrix}1\\1\end{bmatrix}=\vec{v}_1$ $\quad \lambda=-3$ $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ $\quad \begin{matrix}x_1=-x_2\\ x_2=x_2\end{matrix}$ $\quad \vec{v}_2=\begin{bmatrix}-1\\1\end{bmatrix}$

as $t\to\infty,\ \vec{x}\to\begin{bmatrix}0\\0\end{bmatrix},\ \vec{x}=c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-t}+c_2\begin{pmatrix}-1\\1\end{pmatrix}e^{-3t}$



d. $\vec{x}' = \begin{pmatrix}4 & -3\\ 8 & -6\end{pmatrix}\vec{x}$ $\quad \begin{pmatrix}4-\lambda & -3\\ 8 & -6-\lambda\end{pmatrix}$

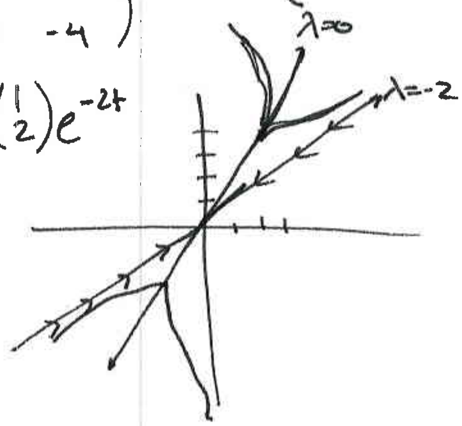
$(4-\lambda)(-6-\lambda)+24=0 \Rightarrow \lambda^2-4\lambda+6\lambda-24+24=0$

$\lambda^2+2\lambda=0 \quad \lambda(\lambda+2)=0 \Rightarrow \lambda=0,\ \lambda=-2$

$\lambda=0$ $\quad \begin{matrix}4x_1=3x_2\\ x_2=x_3\end{matrix}$ $\quad \begin{bmatrix}3\\4\end{bmatrix}=\vec{v}_1$ $\quad \lambda=-2$ $\begin{pmatrix}6 & -3\\ 8 & -4\end{pmatrix}$

$\begin{matrix}2x_1=x_2\\ x_2=x_2\end{matrix}$ $\quad \vec{v}_2=\begin{bmatrix}1\\2\end{bmatrix}$ $\quad \vec{x}=c_1\begin{pmatrix}3\\4\end{pmatrix}+c_2\begin{pmatrix}1\\2\end{pmatrix}e^{-2t}$

as $t\to\infty,\ x\to c_1\begin{bmatrix}3\\4\end{bmatrix}e^{0t}$ or $c_1\begin{bmatrix}3\\4\end{bmatrix}$



e. $\vec{x}' = \begin{pmatrix}2 & 2+i\\ -1 & -1-i\end{pmatrix}\vec{x}$ $\quad \begin{pmatrix}2-\lambda & 2+i\\ -1 & -1-i-\lambda\end{pmatrix}$

$(2-\lambda)(-1-i-\lambda)+2+i=0$

$-\cancel{2}-2i-2\lambda+\lambda+\lambda i+\lambda^2+\cancel{2}+i=0 \Rightarrow \lambda^2+(i-1)\lambda-i=0$

$\lambda=1,-i$

$\lambda=1$ $\begin{pmatrix}1 & 2+i\\ -1 & -2-i\end{pmatrix}$ $\quad \begin{matrix}x_1=-(2+i)x_2\\ x_2=x_2\end{matrix}$ $\quad \vec{v}_1=\begin{bmatrix}-2-i\\1\end{bmatrix}$ $\quad \lambda=-i$ $\begin{pmatrix}2+i & 2+i\\ -1 & -1\end{pmatrix}$ $\quad \begin{matrix}x_1=-x_2\\ x_2=x_2\end{matrix}$ $\quad \begin{bmatrix}1\\1\end{bmatrix}=\vec{v}_2$

6e cont'd.

$$c_1\begin{pmatrix}-2-i\\1\end{pmatrix}e^t + c_2\begin{pmatrix}1\\1\end{pmatrix}(\cos t - i\sin t)$$

$$\begin{pmatrix}-2e^t+\cos t + i(-e^t-\sin t)\\ e^t+\cos t - i\sin t\end{pmatrix} \Rightarrow \vec{x} = c_1\begin{pmatrix}-2e^t+\cos t\\ e^t+\cos t\end{pmatrix} + c_2\begin{pmatrix}-e^t-\sin t\\ -\sin t\end{pmatrix}$$

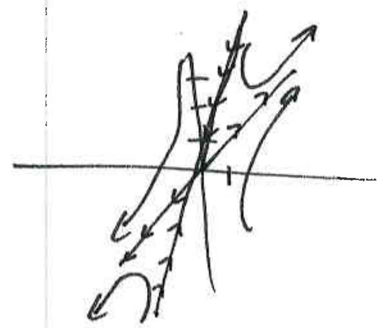
as $t\to\infty$, $\vec{x}\to\infty$

f. $t\vec{x}' = \begin{pmatrix}2 & -1\\ 3 & -2\end{pmatrix}\vec{x}$ $\quad\begin{pmatrix}2-\lambda & -1\\ 3 & -2-\lambda\end{pmatrix} \Rightarrow (-2-\lambda)(2-\lambda)+3=0$

$\lambda^2+2\lambda-2\lambda-4+3=0$

$\lambda^2-1=0 \quad \lambda=\pm 1 \quad \lambda=1 \begin{pmatrix}1 & -1\\ 3 & -3\end{pmatrix} \quad \begin{matrix}x_1=x_2\\ x_2=x_2\end{matrix} \quad \vec{v}_1=\begin{bmatrix}1\\1\end{bmatrix}$

$\lambda=-1 \begin{pmatrix}3 & -1\\ 3 & -1\end{pmatrix} \quad \begin{matrix}3x_1=x_2\\ x_2=x_2\end{matrix} \quad \begin{bmatrix}1\\3\end{bmatrix}=\vec{v}_2$



$\vec{x} = c_1\begin{pmatrix}1\\1\end{pmatrix}t + c_2\begin{pmatrix}1\\3\end{pmatrix}t^{-1}$

as $t\to\infty$, $x\to c_1\begin{pmatrix}1\\1\end{pmatrix}t$

7. $\vec{x}' = \begin{pmatrix}3 & 2 & 4\\ 2 & 0 & 2\\ 4 & 2 & 3\end{pmatrix}\vec{x}$ $\quad\begin{pmatrix}3-\lambda & 2 & 4\\ 2 & -\lambda & 2\\ 4 & 2 & 3-\lambda\end{pmatrix} \Rightarrow (3-\lambda)[-\lambda(3-\lambda)-4]$

$-2[2(3-\lambda)-8]+4[4+4\lambda]=0$

$(3-\lambda)(-3\lambda+\lambda^2-4)-2(6-2\lambda-8)+4(4+4\lambda)=0$

$-9\lambda+3\lambda^2-12+3\lambda^2-\lambda^3+4\lambda-12+4\lambda+16+16+16\lambda=0$

$-\lambda^3+6\lambda^2+15\lambda+8=0 \quad \lambda^3-6\lambda^2-15\lambda-8=0 \Rightarrow (\lambda+1)^2(\lambda-8)=0$

$\lambda=8$, $\lambda=-1$ (repeated)

$\lambda=8 \begin{pmatrix}-5 & 2 & 4\\ 2 & -8 & 2\\ 4 & 2 & -5\end{pmatrix} \Rightarrow \begin{pmatrix}1 & 0 & -1\\ 0 & 1 & -1/2\\ 0 & 0 & 0\end{pmatrix} \Rightarrow \begin{matrix}x_1=x_3\\ x_2=1/2x_3\\ x_3=x_3\end{matrix} \Rightarrow \vec{v}_1=\begin{bmatrix}2\\1\\2\end{bmatrix}$

$\lambda=-1 \begin{pmatrix}4 & 2 & 4\\ 2 & 1 & 2\\ 4 & 2 & 4\end{pmatrix} \Rightarrow \begin{pmatrix}1 & 1/2 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \Rightarrow \begin{matrix}x_1=-1/2x_2-x_3\\ x_2=x_2\\ x_3=x_3\end{matrix} \Rightarrow \vec{x}=\begin{bmatrix}-1\\2\\0\end{bmatrix}x_2+\begin{bmatrix}-1\\0\\1\end{bmatrix}x_3$

$\vec{v}_2=\begin{bmatrix}-1\\2\\0\end{bmatrix} \quad \vec{v}_3=\begin{bmatrix}-1\\0\\1\end{bmatrix}$

$\vec{x} = c_1\begin{pmatrix}-1\\2\\0\end{pmatrix}e^{-t} + c_2 t\begin{pmatrix}-1\\0\\1\end{pmatrix}e^{-t} + c_3\begin{pmatrix}2\\1\\2\end{pmatrix}e^{8t}$

(5)

8a. $\vec{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \vec{x} \quad \vec{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$(5-\lambda)(1-\lambda)+3=0 \Rightarrow 5-6\lambda+\lambda^2+3=0 \Rightarrow \lambda^2-6\lambda+8=0$

$(\lambda-4)(\lambda-2)=0 \quad \lambda=2,4$

$\lambda=2 \quad \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \quad \begin{matrix} 3x_1 = x_2 \\ x_2 = x_2 \end{matrix} \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \vec{v}_1$

$\lambda=4 \quad \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \quad \begin{matrix} x_1 = x_2 \\ x_2 = x_2 \end{matrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{v}_2$

$\vec{x} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} \qquad c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad c_1 = 3/2, \ c_2 = 7/2$

$\vec{x} = -\frac{3}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + \frac{7}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

b. $\vec{x}' = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix} \vec{x} \qquad \begin{pmatrix} -\lambda & 0 & -1 \\ 2 & -\lambda & 0 \\ -1 & 2 & 4-\lambda \end{pmatrix} \Rightarrow (-\lambda)[-\lambda(4-\lambda)] - 1[4-\lambda] = 0$

$(4-\lambda)[\lambda^2 - 1] = 0 \quad \lambda = 4, \ \lambda = \pm 1$

$\lambda=-1 \quad \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} x_1 = x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{matrix} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\lambda=1 \quad \begin{pmatrix} -1 & 0 & -1 \\ 2 & -1 & 0 \\ -1 & 2 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} x_1 = -x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{matrix} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

$\lambda=4 \quad \begin{pmatrix} -4 & 0 & -1 \\ 2 & -4 & 0 \\ -1 & 2 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1/4 \\ 0 & 1 & 1/8 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} x_1 = -1/4\, x_3 \\ x_2 = -1/8\, x_3 \\ x_3 = x_3 \end{matrix} \quad \vec{v}_3 = \begin{bmatrix} -2 \\ -1 \\ 8 \end{bmatrix}$

$\vec{x} = c_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} -2 \\ -1 \\ 8 \end{pmatrix} e^{4t}$

$c_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} -2 \\ -1 \\ 8 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix} \Rightarrow c_1 = 3, \ c_2 = 6, \ c_3 = 1$

$\vec{x} = 3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-t} - 6 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} e^{t} + 1 \begin{pmatrix} -2 \\ -1 \\ 8 \end{pmatrix} e^{4t}$

9a. $\vec{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \vec{x}$

$(3-\lambda)(-1-\lambda)+8=0$

$\lambda^2 - 3\lambda + \lambda - 3 + 8 = 0$

$\lambda^2 - 2\lambda + 5 = 0$

$\begin{pmatrix} 3-(1+2i) & -2 \\ 4 & -1-(1+2i) \end{pmatrix} = \begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}$

$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$

$\frac{4x_1}{4} = \frac{(2+2i)x_2}{4}$

$x_1 = \frac{1+i}{2} x_2$

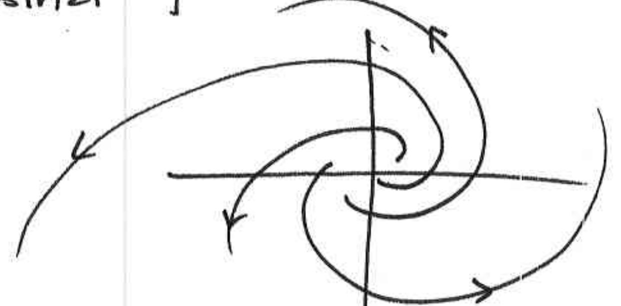
$x_2 = x_2$

$\begin{bmatrix} 1+i \\ 2 \end{bmatrix} = \vec{v}_1, \ \vec{v}_2 = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$

9a cont'd.

$$\begin{bmatrix}1+i\\2\end{bmatrix}e^{(1+2i)t} = e^t\begin{bmatrix}1+i\\2\end{bmatrix}(\cos 2t + i\sin 2t) = e^t\begin{bmatrix}\cos 2t + i\sin 2t + i\cos 2t - \sin 2t\\ 2\cos 2t + 2i\sin 2t\end{bmatrix}$$

$$\vec{x} = c_1 e^t\begin{bmatrix}\cos 2t - \sin 2t\\ 2\cos 2t\end{bmatrix} + c_2 e^t\begin{bmatrix}\sin 2t + \cos 2t\\ 2\sin 2t\end{bmatrix}$$



b. $\vec{x}' = \begin{pmatrix}1 & 2\\ -5 & -1\end{pmatrix}\vec{x}$ $\quad \begin{pmatrix}1-\lambda & 2\\ -5 & -1-\lambda\end{pmatrix}$

$(1-\lambda)(-1-\lambda) + 10 = 0$

$\lambda^2 - 1 + 10 = 0 \qquad \lambda^2 + 9 = 0 \qquad \lambda = \pm 3i$

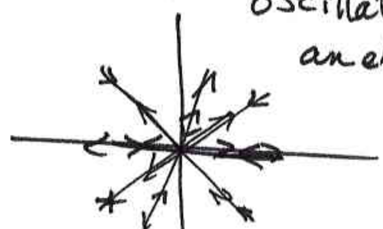
$\begin{pmatrix}1-3i & 2\\ -5 & -1-3i\end{pmatrix} \qquad \dfrac{-5x_1}{-5} = \dfrac{(1+3i)x_2}{-5} \qquad x_1 = \dfrac{1+3i}{-5}x_2 \qquad \begin{bmatrix}1+3i\\-5\end{bmatrix}$

$x_2 = x_2$

$$\begin{bmatrix}1+3i\\-5\end{bmatrix}e^{(\pm 3i)} = \begin{bmatrix}1+3i\\-5\end{bmatrix}(\cos 3t + i\sin 3t) = \begin{pmatrix}\cos 3t + i\sin 3t + 3i\cos 3t - 3\sin 3t\\ -5\cos 3t - 5i\sin 3t\end{pmatrix}$$

$$\vec{x} = c_1\begin{pmatrix}\cos 3t - 3\sin 3t\\ -5\cos 3t\end{pmatrix} + c_2\begin{pmatrix}\sin 3t + 3\cos 3t\\ -5\sin 3t\end{pmatrix}$$

oscillates w/in an ellipse



10. a. $\vec{x}' = \begin{pmatrix}\alpha & 1\\ -1 & \alpha\end{pmatrix}\vec{x}$ $\quad \begin{pmatrix}\alpha-\lambda & 1\\ -1 & \alpha-\lambda\end{pmatrix}$

$(\alpha-\lambda)^2 + 1 = 0 \qquad \alpha - \lambda = \pm i$

$\alpha \pm i = \lambda$ $\quad$ character changes on either side of $\alpha = 0$

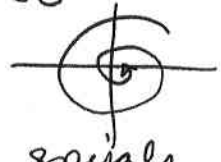
$\alpha > 0$

Spirals outward

$\alpha = 0$

trapped inside ellipse

$\alpha < 0$



spirals inward

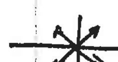
b. $\vec{x}' = \begin{pmatrix}2 & -5\\ \alpha & -2\end{pmatrix}\vec{x}$

$\begin{pmatrix}2-\lambda & -5\\ \alpha & -2-\lambda\end{pmatrix} \Rightarrow (2-\lambda)(-2-\lambda) + 5\alpha = 0$

$\lambda^2 - 4 + 5\alpha = 0 \Rightarrow \lambda^2 + (5\alpha - 4) = 0 \qquad \lambda^2 = \pm\sqrt{4-5\alpha}$

$4 - 5\alpha = 0 \qquad 5\alpha = 4 \qquad \alpha = \frac{4}{5}$ cut off

$\alpha > \frac{4}{5}$ roots are imaginary (pure)



etc. oscillates inside ellipse

$\alpha = \frac{4}{5}$ roots are 0 & repeated

heads off on single trajectory a stable

$\alpha < \frac{4}{5}$ roots are real & distinct forms saddle point

one $\lambda > 0$, one $\lambda < 0$

11. $\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -2x_1 + x_2 \\ x_1 - 2x_2 \end{bmatrix} \Rightarrow \vec{x}'' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

let $x_3 = x_1' \Rightarrow x_1'' = x_3'$ let $x_4 = x_2' \Rightarrow x_2'' = x_4'$

a.
b. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ $\det \begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -2 & 1 & -\lambda & 0 \\ 1 & -2 & 0 & -\lambda \end{bmatrix}$

$\Rightarrow \lambda^4 + 4\lambda^2 + 3 = 0$

$(\lambda^2 + 3)(\lambda^2 + 1) = 0 \quad \lambda = \pm\sqrt{3}i, \ \lambda = \pm i$

$\lambda = i \Rightarrow \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & 0 & i \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\begin{aligned} x_1 &= -ix_4 \\ x_2 &= -ix_4 \\ x_3 &= x_4 \\ x_4 &= x_4 \end{aligned}$ $\begin{bmatrix} -i \\ -i \\ 1 \\ 1 \end{bmatrix} = \vec{v}_1$

$\lambda = \sqrt{3}i \Rightarrow \begin{pmatrix} 1 & 0 & 0 & -\frac{\sqrt{3}}{3}i \\ 0 & 1 & 0 & \sqrt{3}/3\, i \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\begin{aligned} x_1 &= \frac{1}{\sqrt{3}}i x_4 \\ x_2 &= -1/\sqrt{3}\, i x_4 \\ x_3 &= -x_4 \\ x_4 &= x_4 \end{aligned}$ $\begin{bmatrix} i \\ -i \\ -\sqrt{3} \\ \sqrt{3} \end{bmatrix} = \vec{v}_2$

$\begin{bmatrix} -i \\ -i \\ 1 \\ 1 \end{bmatrix}(\cos t + i\sin t) + \begin{bmatrix} i \\ -i \\ -\sqrt{3} \\ \sqrt{3} \end{bmatrix}(\cos(\sqrt{3}t) + i\sin(\sqrt{3}t))$

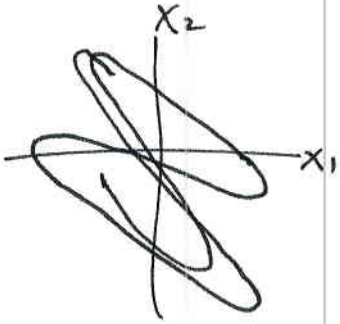
c.
$\vec{x} = c_1\begin{pmatrix} \sin t \\ \sin t \\ \cos t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} -\cos t \\ -\cos t \\ \sin t \\ \sin t \end{pmatrix} + c_3\begin{pmatrix} -\sin(\sqrt{3}t) \\ \sin(\sqrt{3}t) \\ -\sqrt{3}\cos(\sqrt{3}t) \\ \sqrt{3}\cos(\sqrt{3}t) \end{pmatrix} + c_4\begin{pmatrix} \cos(\sqrt{3}t) \\ -\cos(\sqrt{3}t) \\ -\sqrt{3}\sin(\sqrt{3}t) \\ \sqrt{3}\sin(\sqrt{3}t) \end{pmatrix}$

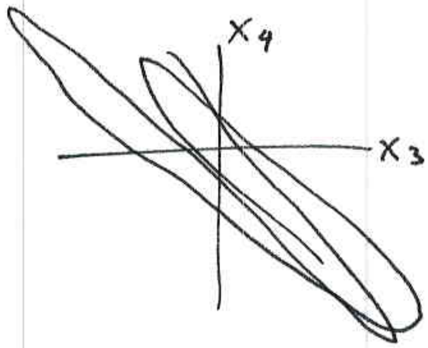
d. $\omega = 1, \omega = \sqrt{3}$ or period $2\pi, \frac{2\pi}{\sqrt{3}}$

e. $y(0) = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$ $c_1\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ -\sqrt{3} \\ \sqrt{3} \end{pmatrix} + c_4\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$ $\begin{aligned} c_1 &= c_3 = 0 \\ c_2 &= -1 \\ c_4 &= -2 \end{aligned}$ $\vec{x} = -1\begin{pmatrix} -\cos t \\ -\cos t \\ \sin t \\ \sin t \end{pmatrix} - 2\begin{pmatrix} \cos(\sqrt{3}t) \\ -\cos(\sqrt{3}t) \\ -\sqrt{3}\sin(\sqrt{3}t) \\ \sqrt{3}\sin(\sqrt{3}t) \end{pmatrix}$

11 cont'd.





f. answers will vary.