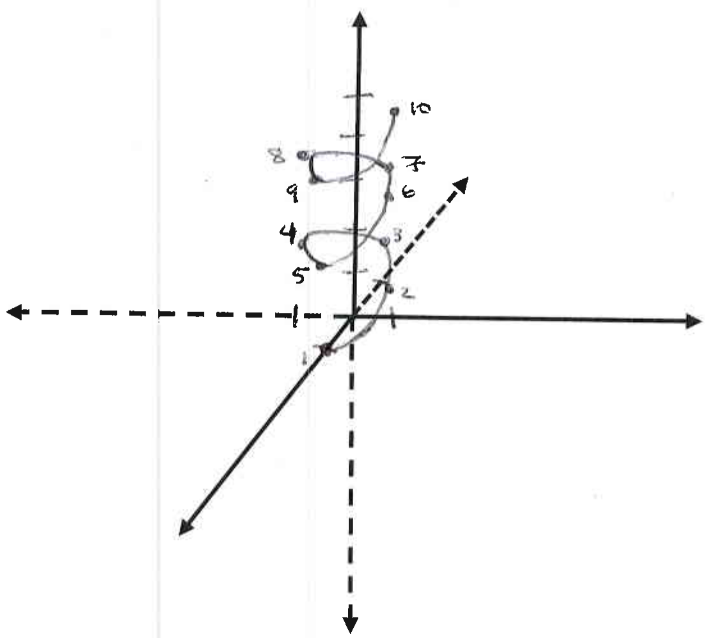
Math 2153, Exam #1, Summer 2015 Name KEY

Instructions: Show all work. Justify answers as completely as possible. If you are asked to prove something, mere computation is not enough. You must explain your reasoning. Be sure to state your conclusion clearly. Incomplete work or justification will not receive full credit. Use exact answers unless specifically asked to round.

1. Plot the space curve given by $\vec{r}(t) = 2\cos t\,\hat{\imath} + 3\sin t\,\hat{\jmath} + \frac{1}{3}t\hat{k}$ on the graph below. Be sure to plot at least 10 points on the curve. (7 points)

t	x	y	z
0	2	0	0
$\pi/2$	0	3	$\pi/6 \approx .52$
π	-2	0	$\pi/3 \approx 1$
$3\pi/2$	0	-3	$\pi/2 \approx 1.57$
2π	2	0	$2\pi/3 \approx 2.1$
$5\pi/2$	0	3	$5\pi/6 \approx 2.62$
3π	-2	0	$\pi \approx 3.14$
$7\pi/2$	0	-3	$7\pi/6 = 3.66$
4π	2	0	$4\pi/3 = 4.12$
$9\pi/2$	0	3	$3\pi/2 \approx 4.71$



2. Use the matrices shown to calculate the given values. (4 points each)

$$A = \begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, C = [4 \quad 1 \quad -3], D = [-2 \quad 5 \quad -1], E = \begin{bmatrix} 4 & 1 & -3 \\ -2 & 1 & 2 \\ 1 & -1 & 5 \end{bmatrix}$$

a. $A\vec{B}$

$$\begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5+8 \\ -10+2 \end{bmatrix} = \begin{bmatrix} 13 \\ -8 \end{bmatrix}$$

b. $\vec{C} \times \vec{D}$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 1 & -3 \\ -2 & 5 & -1 \end{vmatrix} = (-1+15)\hat{\imath} - (-4-6)\hat{\jmath} + (20+2)\hat{k}$$

$$= 14\hat{\imath} + 10\hat{\jmath} + 22\hat{k}$$

c. $\|C\|$ $\sqrt{4^2+1^2+9} = \sqrt{16+1+9} = \sqrt{26}$

d. $\vec{C}\cdot\vec{D}$

$4(-2)+1(5)+(-3)(-1) =$

$-8+5+3=0$

e. $\det(E)$

$\begin{vmatrix} 4 & 1 & -3 \\ -2 & 1 & 2 \\ 1 & -1 & 5 \end{vmatrix} = 4\begin{vmatrix} 1 & 2 \\ -1 & 5 \end{vmatrix} - 1\begin{vmatrix} -2 & 2 \\ 1 & 5 \end{vmatrix} - 3\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = 4(5+2)-(-10-2)-3(2-1)$

$= 4(7)+12 \quad -3 = 37$

3. Find the domain of the functions below. Express each in proper set notation. (4 points each)

a. $\vec{r}(t) = \ln(t-1)\,\hat{\imath} + \sqrt{t}\,\hat{\jmath} + \sin t\,\hat{k}$

$t-1>0$

$t>1 \quad (1,\infty) \longrightarrow \quad (1,\infty)$

$t\geq 0 \quad [0,\infty)$ or $\{t \mid t>1\}$

All reals

b. $f(x,y) = \arcsin(x+y)$ $\quad -1 \leq x+y \leq 1$

$\{(x,y) \mid -1 \leq (x+y) \leq 1\}$

4. Convert the equation $x^2 + y^2 - 3z^2 = 0$ from rectangular to cylindrical coordinates. (4 points)

$r^2 = 3z^2$

$\Rightarrow r = \sqrt{3}z$

$\boxed{\frac{r}{\sqrt{3}} = z}$

5. Convert the equation $z = r^2 \cos^2\theta$ from cylindrical to spherical coordinates. (4 points)

$\rho \cos\varphi = \rho^2 \sin^2\varphi \cos^2\theta$

$\cos\varphi = \rho \sin^2\varphi \cos^2\theta$

$\frac{\cos\varphi}{\sin^2\varphi} \sec^2\theta = \rho$

$\rho = \sec^2\theta \cot\varphi \csc\varphi$

6. Find a parametric equation for the curve created from the intersection of surfaces $z = x^2 + y^2, x + y = 0 \quad x = t$. (5 points)

$X = t$

$t + y = 0$

$y = -t$

$z = (t)^2 + (-t)^2 = t^2 + t^2 = 2t^2$

$r(t) = t\hat{i} - t\hat{j} + 2t^2\hat{k}$

7. Find a parametric (vector-valued) function for the surface given by $4x^2 + y^2 = 16$. (5 points)

$\frac{x^2}{4} + \frac{y^2}{16} = 1 \qquad \begin{matrix} x = 2\cos\theta \\ y = 4\sin\theta \end{matrix} \Rightarrow z$ is free

$\vec{r}(u,v) = 2\cos u\,\hat{i} + 4\sin u\,\hat{j} + v\hat{k}$

8. Find the limits, if they exist, or prove that they do not. You will need to check multiple paths. You may need to use polar or spherical coordinates. (7 points each)

a. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2 y}{x^3 + y^2}$ ① $x = 0$ $\lim\limits_{y\to 0} \frac{0}{y^2} = 0$ ② $y^2 = x^3$, $y = x^{3/2}$

$\lim\limits_{x\to 0} \frac{x^2 x^{3/2}}{x^3 + x^3} = \lim\limits_{x\to 0} \frac{x^3\sqrt{x}}{2x^3} = \lim\limits_{x\to 0} \frac{\sqrt{x}}{2} = 0$

$\lim\limits_{(x,y)\to(0,0)} \frac{x^2 y}{x^3 + y^2} = 0$

b. $\lim_{(x,y)\to(0,0)} \frac{xy^2}{3x^2+2y^2}$ polar $\lim_{r\to 0} \frac{r\cos\theta\, r^2\sin^2\theta}{3r^2\cos^2\theta+2r^2\sin^2\theta} =$

① $x=0$

$\lim_{y\to 0} \frac{0}{2y^2} = 0$

$\lim_{r\to 0} \frac{r^{\cancel{3}}\cos\theta\sin^2\theta}{\cancel{r^2}(3\cos^2\theta+2\sin^2\theta)} = 0$

$\lim_{(x,y)\to(0,0)} \frac{xy^2}{3x^2+2y^2} = 0$

c. $\lim_{(x,y,z)\to(0,0,0)} \arctan\left[\frac{1}{x^2+y^2+z^2}\right]$

$x^2+y^2+z^2=\rho^2$

$\lim_{\rho\to 0} \arctan\left[\frac{1}{\rho^2}\right] = \quad \pi/2$

9. Find the specified integral or derivative for the given vector-valued function. (6 points each)

a. Find $\vec{r}'(t)$ for $\vec{r}(t) = 4\sqrt{t}\,\vec{i} + t^2\sqrt{t}\,\vec{j} + \ln(t^2)\vec{k}$.

$t^{5/2}$ $\quad 2\ln t$

$r'(t) = 4\cdot\frac{1}{2}t^{-1/2}\,\hat{i} + \frac{5}{2}t^{3/2}\hat{j} + \frac{2}{t}\hat{k}$

$= \frac{2}{\sqrt{t}}\hat{i} + \frac{5}{2}t\sqrt{t}\,\hat{j} + \frac{2}{t}\hat{k}$

b. Find $\int_0^1 \vec{r}(t)dt$ for $\vec{r}(t) = e^t\vec{i} + \sec^2 t\,\vec{j} + \frac{1}{t^2+1}\vec{k}$.

$\int_0^1 e^t dt = e^t\Big|_0^1 = e-1$

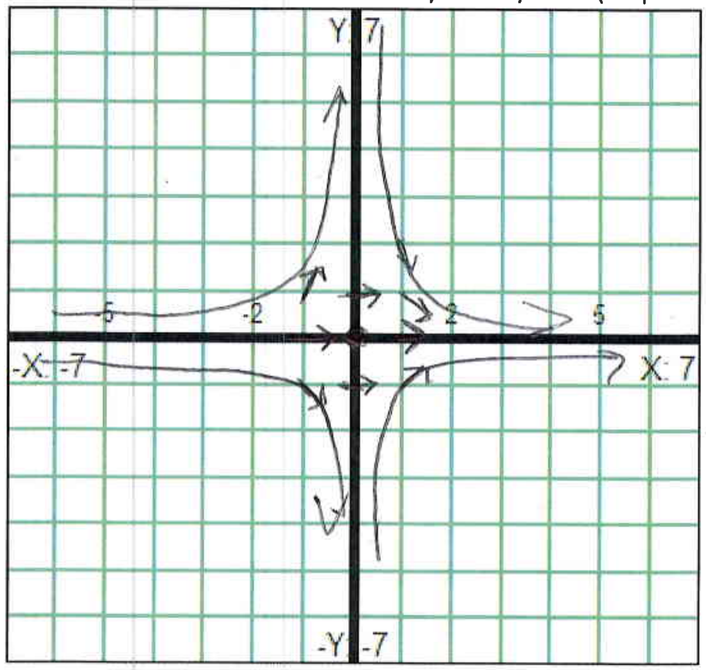
$\int_0^1 \sec^2 t\,dt = \tan t\Big|_0^1 = \quad \tan(1)$

$\int_0^1 \frac{1}{t^2+1}dt = \arctan(1) = \pi/4$

$= (e-1)\hat{i} + \tan 1\,\hat{j} + \pi/4\,\hat{k}$

10. Graph the vector field $\vec{F}(x,y) = (x^2+y^2)\vec{i} - xy\vec{j}$ on the graph below. You must plot at least 10 vectors, then sketch in the rest of the field. Is the field conservative? Why or why not? (10 points)

x	y	x^2+y^2	$-xy$
0	0	0	0
0	1	1	0
1	0	1	0
-1	0	1	0
0	-1	1	0
1	1	2	-1
-1	1	2	1
-1	-1	2	-1
1	-1	2	1
2	1	5	-2



$M = x^2+y^2 \quad N = -xy$

$\frac{\partial M}{\partial y} = 2y$

$\frac{\partial N}{\partial x} = -y$

not equal
So not conservative

11. Evaluate the line integral $\int_C 3(x-y)ds, C: \vec{r}(t) = t\vec{i} + (2-t)\vec{j}, 0 \le t \le 2$. (7 points)

$x = t, \ y = 2-t$

$\vec{r}'(t) = \hat{i} - \hat{j} \quad \|\vec{r}'(t)\| = \sqrt{2}\,dt$

$\int_0^2 3(t-(2-t))\sqrt{2}\,dt$

$3\sqrt{2}\int_0^2 t-2+t\,dt = 3\sqrt{2}\int_0^2 2t-2\,dt$

$= 3\sqrt{2}\left[t^2-2t\right|_0^2 = 3\sqrt{2}\left[4-4-0-0\right] = 0$

12. Find ∇f and $\nabla^2 f$ for $f(x, y, z) = xy\cos z$. (10 points)

$\nabla f = y\cos z\,\hat{\imath} + x\cos z\,\hat{\jmath} + -xy\sin z\,\hat{k}$

$\nabla \cdot \nabla f = 0 + 0 + -xy\cos z = -xy\cos z$

13. Find $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ for $\vec{F}(x, y, z) = \langle \cos xy, \sin xz, \tan y \rangle$. (12 points)

$\vec{\nabla} \cdot \vec{F} = -y\sin xy + 0 + 0 = -y\sin xy$

$\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \cos xy & \sin xz & \tan y \end{vmatrix} = (\sec^2 y - x\cos xz)\hat{\imath} - (0-0)\hat{\jmath} + (z\cos xz + x\sin xy)\hat{k}$

$= \langle \sec^2 y - x\cos xz, 0, z\cos xz + x\sin xy \rangle$

14. For the function $s(x, y) = x^2 + 4xy + y^2 - 4x + 16y + 3$, draw the gradient field below by indicating any curves where the partial derivatives are zero, and at least one vector of the field in each region of the graph. Use that to determine if any critical points are maxima, minima or saddle points. (10 points)

$\frac{\partial s}{\partial x} = 2x + 4y - 4$

$0 = 2x + 4y - 4$

$0 = x + 2y - 2$

$x + 2y = 2$

$\frac{\partial s}{\partial y} = 4x + 2y + 16$

$0 = 2x + y + 8$

$2x + y = -8$

$\nabla f = \langle 2x + 4y - 4, 4x + 2y + 16 \rangle$

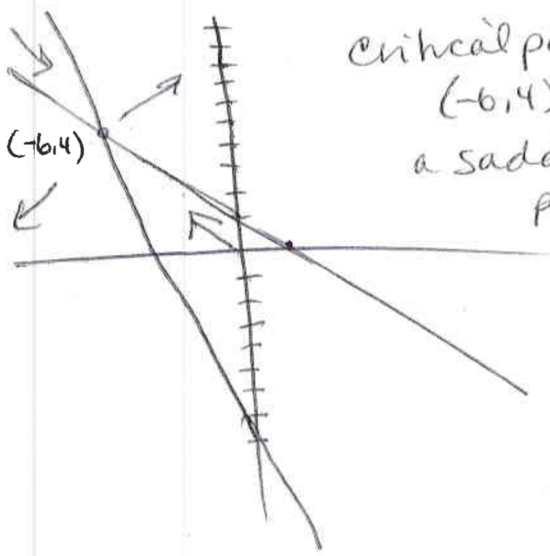
$y = -\frac{1}{2}x + 1$

$y = -2x - 8$

$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & 1 & -8 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & -6 \\ 0 & 1 & 4 \end{array}\right]$

$x = -6$

$y = 4$



critical point (-6,4) is a saddle point

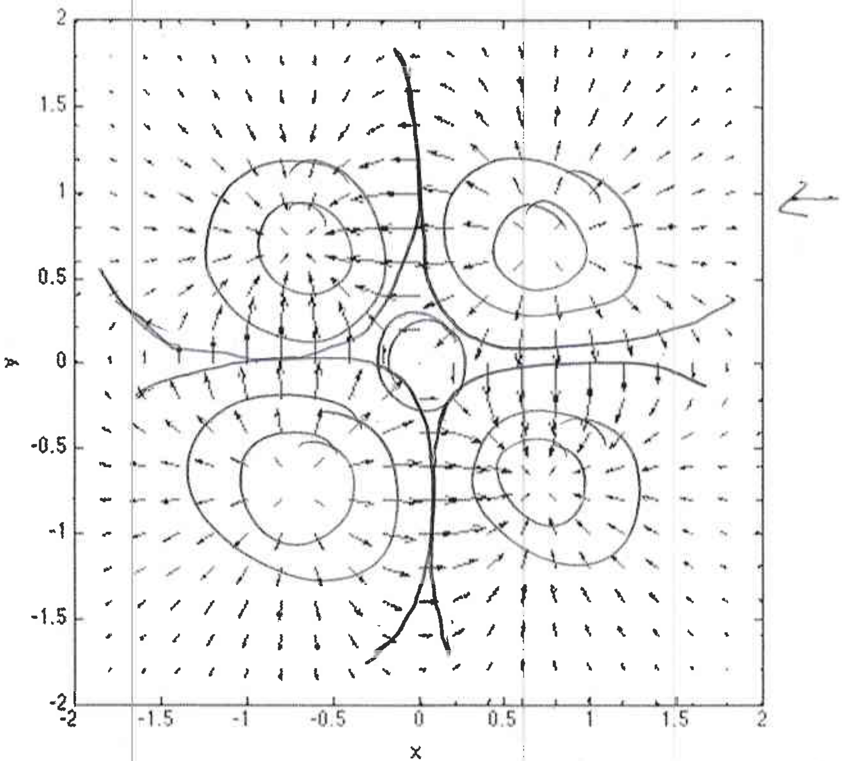
$(0,0) \rightarrow \nabla f = \langle -4, 16 \rangle$

$(0,4) \rightarrow \nabla f = \langle 16-4, 8+16 \rangle = \langle 12, 24 \rangle$

$(-7,5) \rightarrow \nabla f = \langle -14+20-4, -28+10+16 \rangle = \langle 2, -2 \rangle$

$(-6,0) \rightarrow \nabla f = \langle -12-4, -24+16 \rangle = \langle -16, -8 \rangle$

15. For the gradient field shown below, find all the extrema. Label them as maxima, minima, or saddle points. Draw 10 level curves on the graph. (8 points)



(0,0) Saddle point

(.75, -.75) maximum

(.75, .75) minimum

(-.75, -.75) minimum

(-.75, .75) maximum