Math 2415, Final Exam, Summer 2015 Name KEY

Instructions: Show all work. Justify answers as completely as possible. If you are asked to prove something, mere computation is not enough. You must explain your reasoning. Be sure to state your conclusion clearly. Incomplete work or justification will not receive full credit. Use exact answers unless specifically asked to round.

1. Rewrite the differential equation $y'' + 3y' + 4y = 0$ as a system of equations. (8 points)

$y = x_1$ $\quad y' = x_2 = x_1'$ $\quad x_2' + 3x_2 + 4x_1 = 0$

$y'' = x_2'$ $\quad x_1' = x_2$

$$x_1' = x_2$$
$$x_2' = -4x_1 - 3x_2$$

$$\vec{x}\,' = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix} \vec{x}$$

2. Solve the system of linear differential equations given by $\vec{x}\,' = \begin{bmatrix} 7 & -1 \\ 3 & 3 \end{bmatrix} \vec{x}$. Write the solution in the appropriate form. (16 points)

$$(7-\lambda)(3-\lambda) + 3 = 0$$
$$\lambda^2 - 10\lambda + 21 + 3 = 0$$
$$\lambda^2 - 10\lambda + 24 = 0$$
$$(\lambda - 6)(\lambda - 4) = 0$$
$$\lambda = 6, \lambda = 4$$

$\lambda_1 = 6$

$$\begin{bmatrix} 7-6 & -1 \\ 3 & 3-6 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$$

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2$, $x_2 = x_2$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 4$

$$\begin{bmatrix} 7-4 & -1 \\ 3 & 3-4 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$$

$3x_1 - x_2 = 0 \Rightarrow x_1 = \frac{1}{3}x_2$, $x_2 = x_2$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

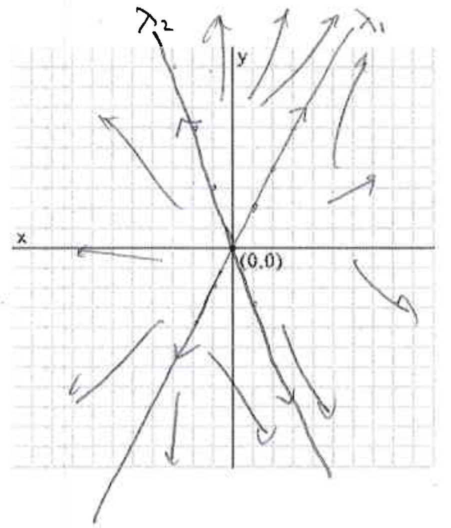
$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{4t}$$

or $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{4t}$$

3. For each of the situations below, determine the properties of the linear system of ODEs. Is the origin an attractor, a repeller, or a saddle point? Sketch the eigenvalues on the graphs provided (if they are real) and plot some sample trajectories. (7 points each)

a. $\lambda_1 = 0.97, \lambda_2 = 0.2, \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

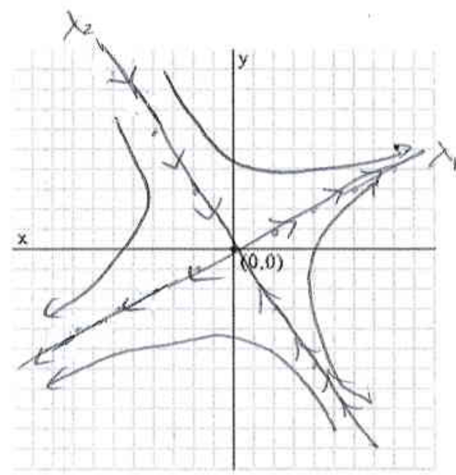
$\lambda > 0$



repeller

b. $\lambda_1 = \frac{7}{5}, \lambda_2 = -2, \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

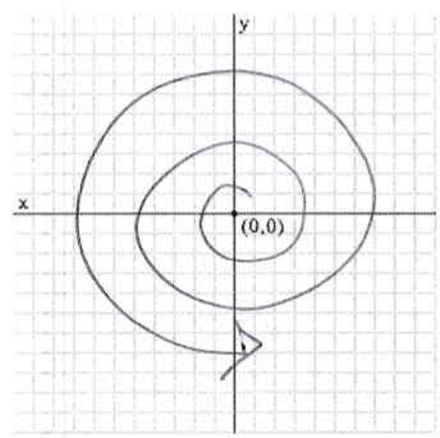
$\lambda > 0$ $\lambda < 0$



Saddle point

c. $\lambda_1 = \frac{1}{2} + \frac{7}{2}i, \lambda_2 = \frac{1}{2} - \frac{7}{2}i.$

$Re\,\lambda > 0$



repeller

4. Find the Fourier series for the function $f(x) = \begin{cases} 3x, & -2 \le x < 0 \\ 1, & 0 \le x < 2 \end{cases}$, $f(x+4) = f(x)$. Be sure to simplify the coefficients as much as possible. (20 points)

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} 3x\,dx + \int_{0}^{2} 1\,dx\right] = \frac{1}{2}\left[\frac{3}{2}x^2\Big|_{-2}^{0} + x\Big|_0^2\right] =$$

$$\frac{1}{2}\left[-\frac{3}{2}(4) + 2\right] = \frac{1}{2}\left[-6+2\right] = \frac{1}{2}[-4] = -2 \qquad \frac{a_0}{2} = -1$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\left[\int_{-2}^{0} 3x\cos\left(\frac{n\pi x}{2}\right)dx + \int_0^2 1\cos\left(\frac{n\pi x}{2}\right)dx\right.$$

$$= \frac{1}{2}\left[\cancel{3x\cdot\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)}^{0} - 3\cdot\frac{2}{n\pi}\cdot\frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{0} + \cancel{\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)}^{0}\Big|_0^2\right] =$$

$$\frac{1}{2}\left[+\frac{12}{n^2\pi^2}(1) - \frac{12}{n^2\pi^2}\cos(-n\pi)\right] = \frac{1}{2}\cdot\frac{12}{n^2\pi^2}\left[1-(-1)^n\right] = \frac{6}{n^2\pi^2}\left[1-(-1)^n\right]$$

$\rightarrow$ n even $\rightarrow 0$
n odd $\rightarrow 2$

$$a_k = \frac{12}{(2k+1)^2\pi^2}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\left[\int_{-2}^{0} 3x\sin\left(\frac{n\pi x}{2}\right)dx + \int_0^2 1\sin\left(\frac{n\pi x}{2}\right)dx\right.$$

$$= \frac{1}{2}\left[3x\cdot\frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) - \cancel{\left(\frac{-6}{n\pi}\right)\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)}^{0}\Big|_{-2}^{0} + \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2\right]$$

$$= \frac{1}{2}\left[0 + \overset{-12}{\frac{6(-2)}{n\pi}}\underbrace{\cos(-n\pi)}_{(-1)^n} - \frac{2}{n\pi}\underbrace{\cos(n\pi)}_{(-1)^n} + \frac{2}{n\pi}(1)\right] =$$

$$\frac{1}{2}\left[\frac{-\overset{7}{\cancel{14}}}{n\pi}(-1)^n + \frac{\overset{1}{\cancel{2}}}{n\pi}\right]$$

$$= \frac{1}{n\pi}\left[1-7(-1)^n\right]$$

n even $\rightarrow -6$
n odd $\rightarrow 8$

$$b_k = \frac{-6}{2k\pi} = \frac{-3}{k\pi} \qquad \sin\left(\frac{2k\pi x}{2}\right) = \sin(k\pi x)$$

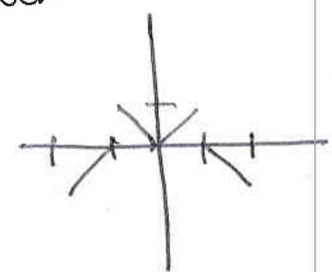
$$c_k = \frac{8}{(2k+1)\pi}$$

$$f(x) = -1 + \sum_{n=1}^{\infty}\left[\frac{12}{(2k+1)^2\pi^2}\cos\left(\frac{(2k+1)\pi x}{2}\right) - \frac{3}{k\pi}\sin(k\pi x) + \frac{8}{(2k+1)\pi}\sin\left(\frac{(2k+1)\pi x}{2}\right)\right]$$

5. For the function below, rewrite the function so that the resulting function is a) even, b) odd. Sketch the graph in each case. State the length of the period. (14 points)

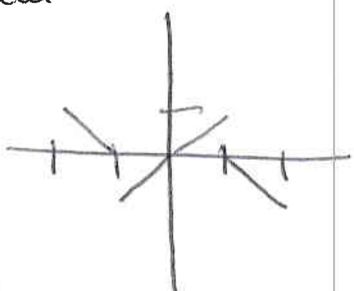
$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ -x + 1, & 1 \le x < 2 \end{cases}$$

a) even



$$f(x) = \begin{cases} x+1 & -2 \le x < 1 \\ |x| & -1 \le x < 1 \\ -x+1 & 1 \le x < 2 \end{cases} \quad \text{or} \quad \begin{cases} x+1 & -2 \le x \le -1 \\ -x & -1 \le x < 0 \\ x & 0 \le x < 1 \\ -x+1 & 1 \le x < 2 \end{cases}$$

b) odd



$$f(x) = \begin{cases} -x-1 & -2 \le x < -1 \\ x & -1 \le x < 1 \\ -x+1 & 1 \le x < 2 \end{cases}$$

6. Under what conditions does the Fourier series contain only sine functions? Under what conditions does it contain only cosine function? When must it contain both types of functions? Explain your reasoning. (12 points)

Fourier series contain only sine functions when the graph is odd. It contains only cosine terms (give-or-take a constant) when the graph is even. It must contain both if it is not symmetric through the origin or along the y-axis.

7. Determine if the partial differential equations below can be solved with the method of separation of variables. (8 points each)

a. $t^2 u_{xx} + u_t = 0$

$u(x,t) = X(x)T(t)$

$u_{xx} = X''T \qquad u_t = XT'$

$t^2 X''T = -XT'$

$\frac{X''}{X} = \frac{-T'}{t^2 T}$ (yes)

b. $u_{xx} + xu_{xt} + u_{tt} = 0$

$u = XT \quad u_{xx} = X''T \quad u_{xt} = X'T' \quad u_{tt} = XT''$

$X''T + xX'T' + XT'' = 0$

(no)

8. Consider a bar 60 cm long that is made of a material for which $\alpha^2 = 4$ and whose ends are insulated. Suppose that the initial temperature is zero except for the interval $20 < x < 40$, where the initial temperature is 385°C. Write the set of equations and initial conditions for the problem with proper notation. You do not need to solve. (7 points)

$\alpha^2 = 4$

$4u_{xx} = u_t$

$$u(x,0) = \begin{cases} 0 & 0 \leq x < 20 \\ 385 & 20 \leq x \leq 40 \\ 0 & 40 \leq x < 60 \end{cases}$$

$u_x(0,t) = 0$

$u_x(60,t) = 0$

9. Solve the second order ordinary differential equations with constant coefficients. (12 points each)
   a. $y'' + y' - 6y = 0$

$$r^2 + r - 6 = 0$$
$$(r-2)(r+3) = 0$$
$$r = 2,\ r = -3$$

$$y = c_1 e^{2t} + c_2 e^{-3t}$$

   b. $y'' + 4y' + 5y = 0$

$$r^2 + 4r + 5 = 0$$
$$r = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

$$y = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$$

10. Determine if the solutions $y_1 = t, y_2 = \sin t, y_3 = \cos t$ form a fundamental set by finding the value of the Wronskian. (12 points)

$$\begin{vmatrix} t & \sin t & \cos t \\ 1 & \cos t & -\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix} = t \begin{vmatrix} \cos t & -\sin t \\ -\sin t & -\cos t \end{vmatrix} - 1 \left[\begin{vmatrix} \sin t & \cos t \\ -\sin t & -\cos t \end{vmatrix} \right] + 0$$

$$= t\left[-\cos^2 t - \sin^2 t\right] - \left[\cancel{-\sin t \cos t + \sin t \cos t}^{\,0}\right]$$

$$= -t\left[\cos^2 t + \sin^2 t\right] = -t$$

Yes, it is a fundamental set for $t \neq 0$

11. Suppose that the solutions to a second order differential equation are $y_1(t) = e^{-t}, y_2(t) = e^{2t}$. If the forcing term on the nonhomogeneous ODE is $F(t) = t^2 \sin t + e^{-2t} + 4\cosh t$, state your initial Ansatz for the method of undetermined coefficients (you do not need to solve for any of the coefficients, just state where you would start). (10 points)

$$Y(t) = (At^2 + Bt + C)\sin t + (Dt^2 + Et + F)\cos t$$
$$+ Ge^{-2t} + He^{t} + Ite^{-t}$$

$4\cosh t = 2e^{-t} + 2e^{t}$

12. Describe when a transient solution exists in a spring problem. (8 points)

a transient solution exists when a spring is damped producing a term or set of terms that decays to zero over time.

13. How do the solutions of a spring system differ if the system is a) undamped, b) underdamped, c) critically damped, d) overdamped? (12 points)

a) undamped $c_1 \sin(\omega t) + c_2 \cos(\omega t)$;
pure imaginary solutions of characteristic equation

b) underdamped $c_1 e^{-at} \sin(\omega t) + c_2 e^{-at} \cos(\omega t)$
complex solutions w/ negative real part.

c) critically damped has repeated roots $c_1 e^{-at} + te^{-at}$

d) overdamped has two distinct real roots
$c_1 e^{-at} + c_2 e^{-bt}$

14. Suppose that a mass of 50 kg stretches a spring 20 cm. Suppose that the mass is attached to a viscous damper with a damping constant of 100 Ns/m. If the mass is pulled down an additional 10 centimeters and then released with an upward velocity of 20 cm/s, find the differential equation and initial conditions to be used to solve for the position of the system. (You do not need to solve the equation, just set it up.) (12 points)

$m = 50$

$\gamma = 100$

$F = 50 \cdot 9.8 = k \cdot .2$

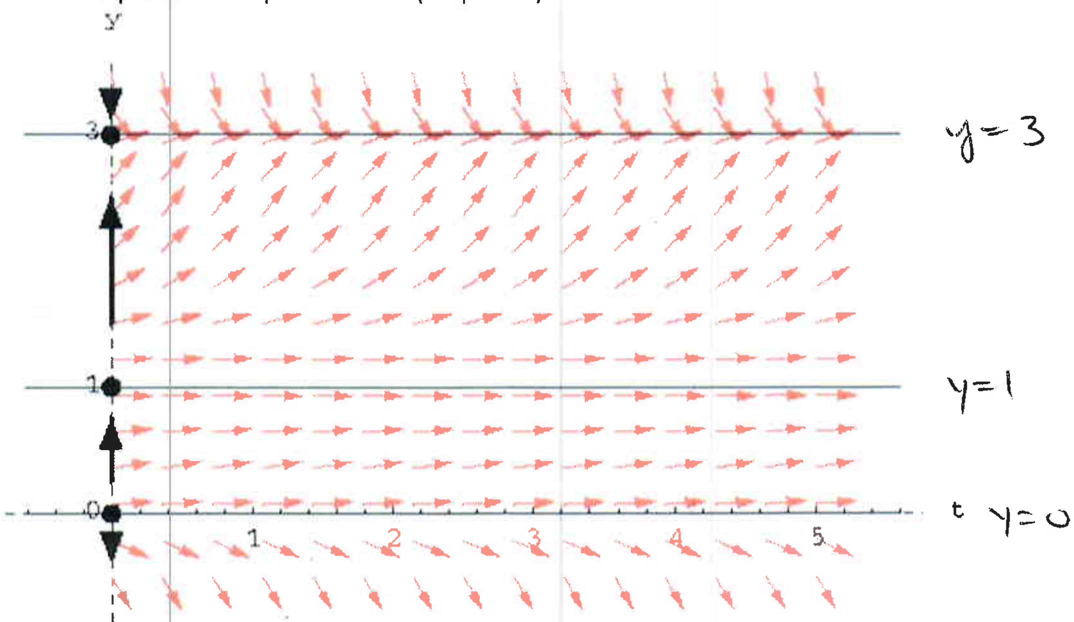
$k = 2450$

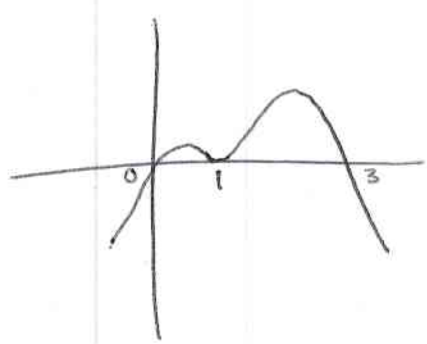
$50y'' + 100y' + 2450y =$

$y(0) = -.1$

$y'(0) = .2$

15. Assuming the equilibrium solutions are integers, use the graph below to sketch the phase portrait of the differential equation that produced the slope field shown here, and write the differential equation that produced it. (12 points)





$\frac{dy}{dt} = -y(y-1)^2(y-3)$ give or take a constant multiplier

16. A tank has pure water flowing into it at 40 L/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 40 L/min. Initially, the tank contains 100 kg of salt in 5000 L of water. Find an equation to model the amount of salt in the tank at any time t. How much salt will there be in the tank after 45 minutes? (15 points)

$A(0) = 100 \text{ kg}$

$$\frac{dA}{dt} = 0 \cdot \frac{40L}{min} - \frac{A}{5000L} \frac{40L}{min}$$

$$\frac{dA}{dt} = -\frac{1}{125} A \implies \frac{dA}{A} = -\frac{1}{125} dt$$

$$\ln A = -\frac{1}{125} t + C \implies A(t) = A_0 e^{-1/125 t}$$

$$A_0 = 100$$

$$A(t) = 100 e^{-1/125 t} \qquad A(45) = 100 e^{-45/100} = 69.77 \text{ kg}$$

17. Use the method of integrating factors to solve the differential equation $y' + \left(\frac{2}{t}\right) y = \frac{\sin t}{t^2}$, $y(0) = 3$. (14 points)

Change to $y(1) = 3$

$$\mu = e^{\int \frac{2}{t} dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$t^2 y' + 2ty = \sin t$$

$$\int (t^2 y)' = \int \sin t \, dt$$

$$\Rightarrow t^2 y = -\cos t + C$$

$$y = \frac{-\cos t}{t^2} + \frac{c}{t^2}$$

Can't solve for $y(0)$. not defined

$$3 = -\frac{\cos(1)}{1^2} + \frac{c}{1^2} \implies c = 3 + \cos(1)$$

$$y(t) = \frac{-\cos t + 3 + \cos(1)}{t^2}$$

18. Classify the following differential equations as a) ordinary or partial, b) order, c) linear or nonlinear. (6 points each)

a. $\frac{d^3y}{dt^3} + y\frac{dy}{dt} + 2y = \cosh t$

third order, ordinary, nonlinear

b. $u_{xx} + u_{yy} = e^{xy}$

partial, linear, order 2
(second order)

19. Solve the equation $2xy'' - y' + 2y = 0$ by series methods. Assume the solution $y = \sum_{n=0}^{\infty} a_n x^n$. Write at least four terms of the series for each solution. (30 points)

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$$

$$2x\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} a_n n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} 2a_n n(n-1)x^{n-1} - \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=1}^{\infty} 2a_{n+1}(n+1)(n)x^n - \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=1}^{\infty} 2a_{n+1}(n+1)(n)x^n - a_1(1)(1) - \sum_{n=1}^{\infty} a_{n+1}(n+1)x^n + 2a_0(1) + \sum_{n=1}^{\infty} 2a_n x^n = 0$$

$$2a_0 - a_1 + \sum_{n=1}^{\infty}\left[2a_{n+1}(n+1)n - a_{n+1}(n+1) + 2a_n\right]x^n = 0$$

$2a_0 - a_1 = 0$

$a_1 = 2a_0$

$2a_{n+1}(n^2+n) - a_{n+1}(n+1) + 2a_n = 0$

$a_{n+1}[2n^2+2n-n-1] = -2a_n$

$$a_{n+1} = \frac{-2a_n}{2n^2+n-1} = \frac{-2a_n}{(2n-1)(n+1)}$$

$$a_2 = \frac{-2a_1}{(2-1)(1+1)} = \frac{-2a_1}{1} = -2a_1 = -4a_0$$

$$a_3 = \frac{-2a_2}{(4-1)(2+1)} = \frac{-2a_2}{9} = \frac{4}{9}a_1 = \frac{8}{9}a_0$$

$$a_4 = \frac{-2a_3}{(6-1)(3+1)} = \frac{-2a_3}{20} = -\frac{1}{10}a_3 = -\frac{8}{90}a_0$$

$$y(x) = a_0\left(1 + 2x - 4x^2 + \frac{8}{9}x^3 - \frac{8}{90}x^4 + \dots\right)$$