Math 2415, Quiz #13, Summer 2015 Name KEY

Instructions: Show all work. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question.

1. Use series solutions methods to find the solution to the ODE $y'' + xy' + 2y = 0, x_0 = 0$. Assume $y = \sum_{n=0}^{\infty} a_n x^n$. Write out at least four terms of each solution.

$y' = \sum_{n=1}^{\infty} a_n \cdot n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} a_n \cdot n(n-1) x^{n-2}$

$a_4 = \frac{-a_2}{3} = \frac{+a_0}{3}$

$a_6 = -\frac{a_4}{5} = -\frac{a_0}{15}$

$a_3 = -\frac{a_1}{2}, \quad a_5 = \frac{-a_3}{4} = +\frac{a_1}{8}, \quad a_7 = \frac{-a_5}{6} = \frac{-a_1}{48}$

$$\boxed{y = a_0\left(1 - x^2 + \tfrac{1}{3}x^4 - \tfrac{1}{15}x^6 + \dots\right) + a_1\left(x - \tfrac{1}{2}x^3 + \tfrac{1}{8}x^5 - \tfrac{1}{48}x^7 + \dots\right)}$$

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + x \sum_{n=1}^{\infty} a_n n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + \sum_{n=1}^{\infty} a_n n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) x^n + \sum_{n=1}^{\infty} a_n n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1) x^n + a_2(2)(1)(1) + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=1}^{\infty} 2a_n x^n + 2a_0(1) = 0$$

$$\sum_{n=1}^{\infty} \left[a_{n+2}(n+2)(n+1) + a_n n + 2a_n\right] x^n + 2a_2 + 2a_0 = 0$$

$\boxed{a_0 = -a_2}$

$a_{n+2}(n+2)(n+1) + (n+2)a_n = 0 \Rightarrow \frac{-a_n \cancel{(n+2)}}{\cancel{(n+2)}(n+1)} = a_{n+2} \Rightarrow \boxed{a_{n+2} = \frac{-a_n}{n+1}}$

2. Use series solutions methods to find the solution to the ODE $x^2 y'' - 2y = 0$. Assume $y = \sum_{n=0}^{\infty} a_n x^n$. Write out at least four terms of each solution.

$(x-1)^2 = x^2 - 2x + 1 \qquad 2(x-1) = 2x - 2$

$(x-1)^2 + 2x - 1 = x^2 \qquad 2(x-1) + 2 = 2x$

$[(x-1)^2 + 2(x-1) + 1]\, y'' - 2y = 0$

$$(x-1)^2 \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} + 2(x-1)\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} + \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} - 2\sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=2}^{\infty} a_n (n)(n-1)(x-1)^n + \sum_{n=2}^{\infty} 2a_n n(n-1)(x-1)^{n-1} + \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} - \sum_{n=0}^{\infty} 2a_n (x-1)^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^n + \sum_{n=1}^{\infty} 2a_{n+1}(n+1)(n)(x-1)^n + \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n - \sum_{n=0}^{\infty} 2a_n (x-1)^n = 0$$

$$\sum_{n=2}^{\infty} \left[a_n n(n-1) + 2a_{n+1}(n+1)n + a_{n+2}(n+2)(n+1) - 2a_n\right](x-1)^n + \cancel{2a_1(1)(0)} + a_2(1)(2) + a_3(3)(2)(x-1) - 2a_0 - 2a_1(x-1) = 0$$

$2a_2 + 6a_3(x-1) - 2a_0 - 2a_1(x-1) = 0$

$2a_2 = 2a_0 \Rightarrow a_0 = a_2$

$6a_3 = 2a_1 \Rightarrow a_3 = \frac{1}{3}a_1$

$a_n n(n-1) + 2a_{n+1}(n)(n+1) + a_{n+2}(n+2)(n+1) - 2a_n = 0$

$$a_{n+2} = \frac{[2 - n(n-1)]a_n - 2a_{n+1} n(n+1)}{(n+2)(n+1)}$$

$$a_4 = \frac{(2 - 2(1))a_2 - 2a_3 2(3)}{4 \cdot 3} = -a_3 = -\frac{1}{3}a_1$$

$$a_5 = \frac{[2 - 3(2)]a_3 - 2a_4 3(4)}{5 \cdot 4} = \frac{-4a_3 - 24a_4}{20} = \frac{\frac{4}{3}a_1 - \frac{24}{3}a_1}{20} = \frac{-\frac{20}{3}a_1}{20} = -\frac{1}{3}a_1$$

$$y = a_0\left(1 + (x-1)^2\right) + a_1\left((x-1) + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{3}(x-1)^4 - \tfrac{1}{3}(x-1)^5 + \dots\right)$$