Math 2415, Quiz #9, Summer 2015 Name KEY

Instructions: Show all work. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question.

1. Find the eigenvalues and eigenfunctions of the equation $y'' + y' + \lambda y = 0, y(0) = 0, y(1) = 0$.

$r^2 + r + \lambda = 0$

$r = \frac{-1 \pm \sqrt{1-4\lambda}}{2}$

3 condetions to check

$1-4\lambda = 0$, $1-4\lambda < 0$, $1-4\lambda > 0$

① $1 = 4\lambda \Rightarrow$

$\frac{1}{4} = \lambda$

$\Rightarrow r = -\frac{1}{2}$ repeated

$y = c_1 e^{-1/2 t} + c_2 t e^{-1/2 t}$

$y(0) = 0 = c_1 + c_2(0)$

$c_1 = 0$

$y(1) = 0 = c_2 (1) e^{-1/2 t}$

$c_2 = 0$

trivial solution

② $1-4\lambda > 0$

$1 > 4\lambda$

$1/4 > \lambda$ or $\lambda < 1/4$

$r = \frac{-1 \pm \sqrt{1-4\lambda}}{2}$ (real solutions)

Call $a = -\frac{1}{2} + \frac{\sqrt{1-4\lambda}}{2}$, $b = -\frac{1}{2} - \frac{\sqrt{1-4\lambda}}{2}$

$y = c_1 e^{at} + c_2 e^{bt}$

$y(0) = 0 = c_1 e^0 + c_2 e^0$

$0 = c_1 + c_2 \Rightarrow c_1 = -c_2$

$y(1) = 0 = c_1 e^a + c_2 e^b$

$0 = c_1 e^a - c_1 e^b = c_1(e^a - e^b)$

$\Rightarrow c_1 = 0$ since $a \neq b$

③ $1-4\lambda < 0 \Rightarrow \lambda > 1/4$

$r = \frac{-1 \pm \sqrt{1-4\lambda}}{2}$ imaginary

call $1-4\lambda = 4a^2$

$r = -\frac{1}{2} \pm ai$

$y = c_1 e^{-1/2 t} \cos at + c_2 e^{-1/2 t} \sin at$

$0 = y(0) = c_1 (1)(1) + c_2 (1)(0) \Rightarrow c_1 = 0$

$y(1) = 0 = c_2 e^{-1/2} \sin(a)$

c_2 not 0 if $a = n\pi$

$1-4\lambda = 4n^2\pi^2$

$4\lambda = 1-4n^2\pi^2$

$\lambda = \frac{1-4n^2\pi^2}{4}$ and $\lambda > 1/4$

$y = c_2 e^{-1/2 t} \sin(n\pi t)$

2. Find the eigenvalues and eigenfunctions of the equation $y'' + \lambda y' + \lambda y = 0, y(0) = 0, y(2) = 0$.

$r^2 + \lambda r + \lambda = 0$

$r = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\lambda}}{2}$

③ conditions

$\lambda^2 - 4\lambda = 0$

$\lambda(\lambda-4) = 0$

$\lambda = 0, \lambda = 4$

$\lambda^2 - 4\lambda < 0$

$\lambda^2 - 4\lambda > 0$

① $\lambda = 0, \lambda = 4$

if $\lambda = 0$ $y'' = 0$

$y(t) = At + B$

$y(0) = 0 = A(0) + B$

$\Rightarrow B = 0$

$y(2) = 0 \Rightarrow A(2) = 0$

$\Rightarrow A = 0$

trivail solution

if $\lambda = 4$ $y'' + 4y' + 4 = 0$

$r = -2$

$y(t) =$

$c_1 e^{-2t} + c_2 t e^{-2t}$

$y(0) = c_1 + c_2(0) \Rightarrow 0$

$c_1 = 0$

$y(2) = c_2(2) e^{-4} = 0$

$\Rightarrow c_2 = 0$

② $\lambda(\lambda-4) < 0$

+ − +

0 4

$0 < \lambda < 4$

Call $\lambda^2 - 4\lambda = 4a^2$

$-\frac{\lambda}{2} \pm ai = r$

$y = c_1 e^{-\lambda/2 t} \cos(at) + c_2 e^{-\lambda/2 t} \sin(at)$

$y(0) = 0 = c_1 (1)(1) + c_2 (1)(0)$

$\Rightarrow c_1 = 0$

$y(2) = 0 = c_2 e^{-\lambda} \sin(2a)$

$\sin(2a) = 0 \Rightarrow a = \pi/2$

$\lambda^2 - 4\lambda = 4(\pi^2/4) = \pi^2$

$\lambda = \frac{4 \pm \sqrt{16 - 4\pi^2}}{2}$ $y(t) = c_2 e^{-\lambda/2 t} \sin(at)$

λ is imaginary

no real values of λ satisfy condition

③ $\lambda(\lambda-4) > 0$

$\lambda^2 - 4\lambda =$

$\lambda < 0$ or $\lambda > 4$

$-\lambda/2 + \frac{\sqrt{\lambda^2-4\lambda}}{2} = a$

$-\lambda/2 - \frac{\sqrt{\lambda^2-4\lambda}}{2} = b$

$y = c_1 e^{at} + c_2 e^{bt}$

$y(0) = 0 = c_1 + c_2 \Rightarrow$

$c_1 = -c_2$

$y(2) = 0 = c_1 e^{2a} + c_2 e^{2b}$

$= c_1 e^{2a} - c_1 e^{2b}$

$= c_1(e^{2a} - e^{2b}) \Rightarrow c_1 = 0$

since $e^{2a} \neq e^{2b}$ since $a \neq b$

trivail solution