Math 2568, Exam #1, Part I, Summer 2015 Name KEY

Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. When you are finished with this portion of exam, get Part II.

1. Write the system of equations $\begin{cases} 3x_1 + \ x_2 = 5 \\ 2x_1 - 5x_2 \ = 11 \end{cases}$ as a) a vector equation, b) a matrix equation, c) an augmented matrix. (8 points)

a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ -5 \end{bmatrix} x_2 = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 1 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$

c) $\left[\begin{array}{cc|c} 3 & 1 & 5 \\ 2 & -5 & 11 \end{array}\right]$

2. Row reduce the system to obtain the solution $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. (8 points)

$\frac{1}{3}R_1 \to R_1$ $\left[\begin{array}{cc|c} 1 & 1/3 & 5/3 \\ 2 & -5 & 11 \end{array}\right]$ $-2R_1 + R_2 \to R_2$ $\left[\begin{array}{cc|c} 1 & 1/3 & 5/3 \\ 0 & -17/3 & 23/3 \end{array}\right]$

$-2/3 - 15/3 = -17/3$

$-10/3 + \frac{33}{3} = \frac{23}{3}$

$-\frac{3}{17} R_2 \to R_2$ $\left[\begin{array}{cc|c} 1 & 1/3 & 5/3 \\ 0 & 1 & -23/17 \end{array}\right]$

$-1/3 R_2 + R_1 \to R_1$ $\left[\begin{array}{cc|c} 1 & 0 & 36/17 \\ 0 & 1 & -23/17 \end{array}\right]$

$5/3 \cdot \frac{17}{17} = \frac{85}{51} + \frac{23}{51} = \frac{108}{51} = \frac{36}{17}$

3. Determine if each statement is True or False. (2 points each)

a. T F Two matrices are row equivalent if they have the same number of rows.

b. T F Two fundamental questions about linear systems is about existence and uniqueness.

c. T F Both $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are matrices in echelon form.

d. T F The echelon form of a matrix is always unique.

e. T F If two points corresponding to two vectors lie on the same line, then the vectors they represent are linearly independent.

f. T F The $span\{\vec{u}, \vec{v}\}$ is the plane containing all the vectors that are linear combinations of $\vec{u}$ and $\vec{v}$.

g. T F The equation $A\vec{x} = \vec{b}$ is inconsistent if the augmented matrix representing the system has a pivot in every row.

h. T F The solution to the system $A\vec{x} = \vec{b}$ is of the form $\vec{x} = \vec{p} + t\vec{v}$ where $\vec{v}$ is any solution to the system $A\vec{x} = \vec{0}$.

i. T F A homogeneous systems of equations can never be inconsistent.

j. T F Matrices of the form $\begin{bmatrix} a & b \\ 1 & d \end{bmatrix}$ is a subspace of $M_{2\times2}$.

$\vec{0}$ not in space

k. T F The function $f(x) = 0$ is a subspace of P_n.

l. T F R^5 is isomorphic to a subspace of R^6.

m. T F If two spaces have the same number of basis vectors, then then are isomorphic.

n. T F The pivot columns of a matrix are always linearly independent.

o. T F The column space of an $m \times n$ matrix is a subspace of R^n.

p. T F A linear transformation defined by a 4x6 matrix can be onto, but it cannot be one-to-one. 4 pivots for onto;

q. T F A set of vectors are linearly independent if the set does not contain the zero vector.

r. T F A vector space has infinite dimensions if there is no finite basis for the space.

s. (T) F The kernel of a matrix is a subspace of the domain of the matrix.

t. T (F) Two matrices, B which is $m \times n$ and C which is $p \times q$, produce a defined product BC when $m = p$.

4. Determine if the following sets are subspaces. Be sure to check all the necessary conditions or find a counterexample. (7 points each)

a. $V = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a + b = 0 \right\}$.

if $a=b=0$ $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in space

$\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}$ $c+d=0$ $\Rightarrow$ $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$ and does $(a+c)+(b+d)=0$?

yes, since $(a+b)+(c+d)=0$

and $k\begin{bmatrix} a \\ b \end{bmatrix}$ has $ka+kb=0$ since $k(a+b)=0$

it is a subspace

b. The set of all polynomials of the form $p(t) = a + bt + ct^6$.

if $a=b=c=0$ $p(t)=0$ $\vec{0}$ in space

$p(t)$, $q(t) = d + et + ft^6$ $p(t)+q(t) = \underbrace{(a+d)}_{\text{real}} + \underbrace{(b+e)}_{\text{real}}t + \underbrace{(f+c)}_{\text{real}}t^6$ in space

and

$kp(t) = \underbrace{(ka)}_{\text{real}} + \underbrace{(kb)}_{\text{real}}t$ and $\underbrace{(c+f)}_{\text{real}}t^6$.

it is a subspace

5. Determine if the transformation $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 5x_2^2 \\ x_1 + 2x_3 \\ 7x_3 - 4x_2 \end{bmatrix}$ is linear or not. If it is, prove it. If it is not, find a counterexample. (8 points)

it is not linear

$3T\left(\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}\right) = 3\begin{bmatrix} 0-5(9) \\ 0+2(0) \\ 0-4(3) \end{bmatrix} = 3\begin{bmatrix} -45 \\ 0 \\ -12 \end{bmatrix} = \begin{bmatrix} -135 \\ 0 \\ -36 \end{bmatrix}$ These are not equal.

$T\left(3\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 9 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0-5(81) \\ 0-0 \\ 0-4(9) \end{bmatrix} = \begin{bmatrix} -405 \\ 0 \\ -36 \end{bmatrix}$

Counterexamples will vary

6. Consider the following matrices: $A = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}, C = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 0 & -1 \\ -1 & 3 & 1 \end{bmatrix}$, find each of the following matrices or say that they are undefined. If they are undefined, explain why. (7 points each)

a. $A^T + 3I_2$

$A^T = \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}$ $\quad 3I_2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ -2 & 4 \end{bmatrix}$

b. BC

$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}}_{2\times3}\underbrace{\begin{bmatrix} 1 & -1 & 2 \\ 4 & 0 & -1 \\ -1 & 3 & 1 \end{bmatrix}}_{3\times3} = \begin{bmatrix} 1+4 & -1 & 2-1 \\ 2-4-3 & -2+9 & 4+1+3 \end{bmatrix}$

$\begin{bmatrix} 5 & -1 & 1 \\ -5 & 7 & 8 \end{bmatrix}$

7. Find the inverse of the matrix $A = \begin{bmatrix} -3 & 1 \\ 2 & 5 \end{bmatrix}$. Use it to solve the system $\begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 18 \end{bmatrix}$. (10 points)

$\det A = -3-4 = -7$

$\frac{1}{-7}\begin{bmatrix} 1 & -1 \\ -4 & -3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -1 & 1 \\ 4 & 3 \end{bmatrix}$

$\frac{1}{7}\begin{bmatrix} -1 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -10 \\ 18 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 10+18 \\ -40+54 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 28 \\ 14 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

8. List 10 statements equivalent is "a nxn matrix A is invertible" from the Invertible Matrix Theorem. (10 points)

answers will vary

- the columns of A are linearily independent
- The matrix reduces to the identity
- the matrix has n pivots
- The Column space of $A = \mathbb{R}^n$
- the null space of A is $\{\vec{0}\}$
- $A\vec{x} = 0$ has only The trivial solution
- the dimensions of the Column space is n
- the linear transformation A is onto
- the linear transformation A is one-to-one
- there exists a matrix C so that $AC = I$
- there exists a matrix D so that $DA = I$
- The determinant of $A \neq 0$
- A^T is invertible.

Math 2568, Exam #1, Part II, Summer 2015 Name ____________________

Instructions: Show all work. You **may** use a calculator on this portion of the exam. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question. Provide explanations where requested.

1. Find the nullspace of the system $\begin{cases} x_1 + 2x_2 - 3x_3 - 2x_4 + x_5 + 2x_6 = 0 \\ 4x_1 - x_2 + x_5 - x_6 = 0 \\ 2x_1 - x_3 + 2x_4 + 4x_6 = 0 \end{cases}$. (12 points)

$$\text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -8/3 & 1 & -4 \\ 0 & 1 & 0 & -32/3 & 3 & -15 \\ 0 & 0 & 1 & -22/3 & 2 & -12 \end{bmatrix}$$

free

$x_1 - 8/3 x_4 + x_5 - 4x_6 = 0$
$x_2 - 32/3 x_4 + 3x_5 - 15x_6 = 0$
$x_3 - 22/3 x_4 + 2x_5 - 12x_6 = 0$
$x_4 = x_4$
$x_5 = x_5$ free
$x_6 = x_6$

$x_1 = 8/3 x_4 - x_5 + 4x_6$
$x_2 = 32/3 x_4 - 3x_5 + 15x_6$
$x_3 = 22/3 x_4 - 2x_5 + 12x_6$
$x_4 = x_4$
$x_5 = x_5$
$x_6 = x_6$

$$\Rightarrow \vec{x} = \begin{bmatrix} 8/3 \\ 32/3 \\ 22/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -1 \\ -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_5 + \begin{bmatrix} 4 \\ 15 \\ 12 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_6$$

$$\text{Nul } A = \left\{ \begin{bmatrix} 8 \\ 32 \\ 22 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 15 \\ 12 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

2. Determine if the following sets of vectors are linearly independent. Then determine if they form a basis for the specified space. Explain your reasoning. (7 points each)

a. $\left\{ \begin{bmatrix} 1 \\ 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}, R^5$

linearly independent
2 vectors / not multiples

they do not span $\mathbb{R}^5$ so they are not a basis for $\mathbb{R}^5$

b. $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}, R^2$

linearly independent
2 vectors / not multiples

they do span $\mathbb{R}^2$ so they do form a basis for $\mathbb{R}^2$

c. $\{1-2t^2, 6-2t, 4t+t^2\}, P_2$ P_2 is isomorphic to $\mathbb{R}^3$

$\begin{bmatrix}1\\0\\-2\end{bmatrix}$ $\begin{bmatrix}6\\-2\\0\end{bmatrix}$, $\begin{bmatrix}0\\4\\1\end{bmatrix}$

rref $\Rightarrow$ $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ pivot in every column so they are independent
pivot in every row so the span $\mathbb{R}^3$

$\Rightarrow$ consequently, these vectors do form a basis for P_2

3. Suppose matrix A is a 9×6 matrix with 4 pivot columns. Determine the following. (12 points)

dim Col A = 4 dim Nul A = 2

dim Row A = 4 If Col A is a subspace of R^m, then m = 9

Rank A = 4 If Nul A is a subspace of R^n, then n = 6

4. If a basis for R^3 is $B = \left\{\begin{bmatrix}1\\-1\\3\end{bmatrix}, \begin{bmatrix}3\\0\\-2\end{bmatrix}, \begin{bmatrix}4\\-3\\0\end{bmatrix}\right\}$, and given $[\vec{x}]_B = \begin{bmatrix}3\\1\\-2\end{bmatrix}$, find $\vec{x}$ in the standard basis. (8 points)

$$P_B = \begin{bmatrix}1&3&4\\-1&0&-3\\3&-2&0\end{bmatrix}$$

$$\vec{x} = P_B[\vec{x}]_B = \begin{bmatrix}1&3&4\\-1&0&-3\\3&-2&0\end{bmatrix}\begin{bmatrix}3\\1\\-2\end{bmatrix} = \begin{bmatrix}3\\-3\\9\end{bmatrix} + \begin{bmatrix}3\\0\\-2\end{bmatrix} + \begin{bmatrix}-8\\6\\0\end{bmatrix} = \begin{bmatrix}1\\3\\7\end{bmatrix}$$

5. If a vector in the standard basis is $\vec{x} = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}$, find its representation in the basis in problem #4.

(9 points)

$$P_B = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 0 & -3 \\ 3 & -2 & 0 \end{bmatrix}$$

$$P_B [x]_B = \vec{x}$$

$$P_B^{-1} \vec{x} = [\vec{x}]_B$$

$$P_B^{-1} = \begin{bmatrix} 6/25 & 8/25 & 9/25 \\ 4/25 & 12/25 & 1/25 \\ -2/25 & -11/25 & -3/25 \end{bmatrix}$$

$$P_B^{-1} \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

6. Consider the basis $C = \left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right\}$, and the vector $[\vec{x}]_C = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$. Find the representation of the vector in the basis B in problem #4. (10 points)

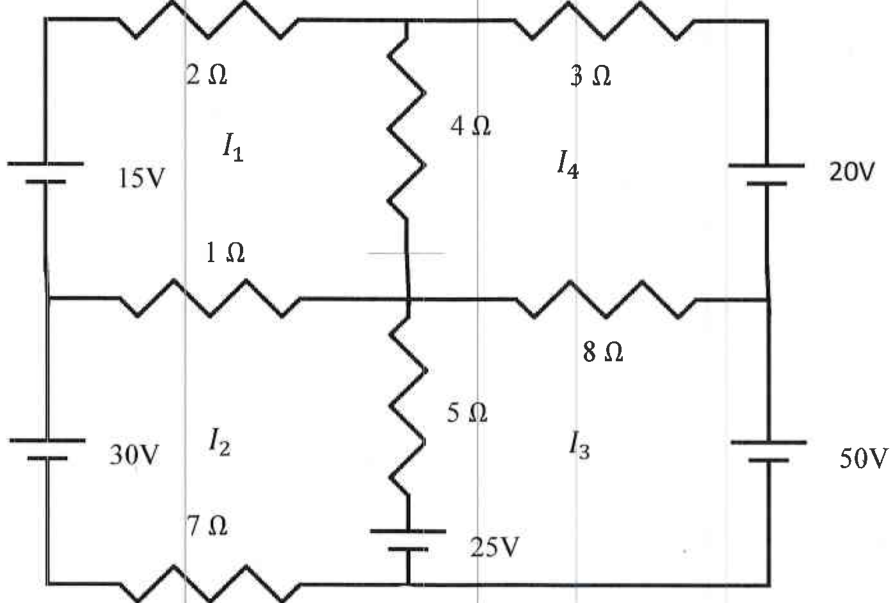
$$P_C [\vec{x}]_C = P_B [\vec{x}]_B$$

$$P_B^{-1} P_C [\vec{x}]_C = [\vec{x}]_B$$

above

$$P_B^{-1} \begin{bmatrix} 1 & -1 & 4 \\ -1 & 2 & -1 \\ -2 & 3 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 199/25 \\ 111/25 \\ -108/25 \end{bmatrix} \Rightarrow \begin{bmatrix} 7.96 \\ 4.44 \\ -4.32 \end{bmatrix}$$

7. Write a matrix to determine the loop currents and use your calculator to solve the system. Round your answers to two decimal places. (14 points)



$(2+1+4)I_1 - I_2 - 4I_4 = -15$ $\Rightarrow$ $7I_1 - I_2 \quad -4I_4 = -15$

$-I_1 + (1+7+5)I_2 - 5I_3 = -30+25$ $\qquad -I_1 + 13I_2 - 5I_3 = -5$

$-5I_2 + (5+8)I_3 - 8I_4 = -25+50$ $\qquad -5I_2 + 13I_3 - 8I_4 = 25$

$-4I_1 - 8I_3 + (3+4+8)I_4 = +20$ $\qquad -4I_1 \quad -8I_3 + 15I_4 = 20$

$$\left[\begin{array}{cccc|c} 7 & -1 & 0 & -4 & -15 \\ -1 & 13 & -5 & 0 & -5 \\ 0 & -5 & 13 & -8 & 25 \\ -4 & 0 & -8 & 15 & 20 \end{array}\right] \Rightarrow$$

$$\vec{I} = \begin{bmatrix} .492 \\ 1.63 \\ 5.14 \\ 4.20 \end{bmatrix}$$