Math 2568, Final Exam, Part I, Summer 2015 Name KEY

Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. When you are finished with this portion of exam, get Part II.

1. An orthogonal basis for R^2 is $\left\{\begin{bmatrix}-1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$. Find an expression for $\vec{x} = \begin{bmatrix}4\\-5\end{bmatrix}$ in this basis. Do not use inverse matrices (use projections). (8 points)

$$c_1 = \frac{-1(4)+2(-5)}{(-1)^2+2^2} = \frac{-4-10}{5} = -\frac{14}{5}$$

$$c_2 = \frac{2(4)+1(-5)}{2^2+1^2} = \frac{8-5}{5} = \frac{3}{5}$$

$$[\vec{x}]_B = \begin{bmatrix}-14/5\\3/5\end{bmatrix}$$

2. For the matrix $A = \begin{bmatrix}6 & 2\\-8 & -2\end{bmatrix}$, find the eigenvalues and eigenvectors of the matrix. (10 points)

$$(6-\lambda)(-2-\lambda)+16=0$$
$$\lambda^2-4\lambda-12+16=0$$
$$\lambda^2-4\lambda+4=0$$
$$(\lambda-2)(\lambda-2)=0$$
$$\lambda=2$$

$$\lambda=2 \quad \begin{bmatrix}6-2 & 2\\-8 & -2-2\end{bmatrix} = \begin{bmatrix}4 & 2\\-8 & -4\end{bmatrix}$$

$$4x_1+2x_2=0$$
$$x_1 = -\tfrac{1}{2}x_2$$
$$x_2 = x_2$$

$$\vec{v}_1 = \begin{bmatrix}-1\\2\end{bmatrix}$$

there is no second eigenvector for this repeated root.

3. A basis for R^3 is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. Use Gram-Schmidt to find an orthogonal basis for the space.

(15 points) $\qquad \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$

$$\vec{b}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{b}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{b}_1}{\vec{b}_1 \cdot \vec{b}_1} \vec{b}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix} - \frac{1-2-1+3}{1+4+1+1} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{7} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - 1/7 \\ -1 - 2/7 \\ 1 + 1/7 \\ 3 - 1/7 \end{bmatrix} = \begin{bmatrix} 6/7 \\ -9/7 \\ 8/7 \\ 20/7 \end{bmatrix}$$

Okay to scale since its a basis vector $\begin{bmatrix} 6 \\ -9 \\ 8 \\ 20 \end{bmatrix} = \vec{b}_2$

$$\vec{b}_3 = \overset{\vec{v}_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}} - \frac{\vec{v}_3 \cdot \vec{b}_1}{\vec{b}_1 \cdot \vec{b}_1} \vec{b}_1 - \frac{\vec{v}_3 \cdot \vec{b}_2}{\vec{b}_2 \cdot \vec{b}_2} \vec{b}_2 =$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1+2-1+0}{1+4+1+1} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} - \frac{6-9+8+0}{36+81+64+400} \begin{bmatrix} 6 \\ -9 \\ 8 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \underset{\uparrow \frac{166}{581}}{\frac{2}{7}} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} - \frac{5}{581} \begin{bmatrix} 6 \\ -9 \\ 8 \\ 20 \end{bmatrix}$$

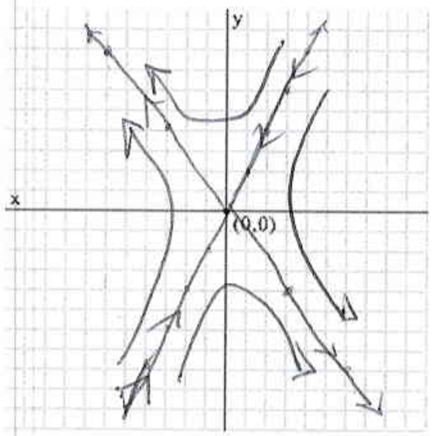
$$581 = 83 \times 7$$

$$= \begin{bmatrix} 1 - 166/581 - \frac{30}{581} \\ 1 - 332/581 + 45/581 \\ 1 + 166/581 - 40/581 \\ 0 - 166/581 - 100/581 \end{bmatrix} = \begin{bmatrix} 385/581 \\ 294/581 \\ 707/581 \\ -266/581 \end{bmatrix} \begin{matrix} = 55/83 \\ = 42/83 \\ = 101/83 \\ = -38/83 \end{matrix} \Rightarrow \begin{bmatrix} 55 \\ 42 \\ 101 \\ -38 \end{bmatrix} = \vec{b}_3$$

4. For each of the situations below, determine the properties of the discrete dynamical system. Is the origin an attractor, a repeller, or a saddle point? Sketch the eigenvalues on the graphs provided (if they are real) and plot some sample trajectories. (6 points each)

a. $\lambda_1 = 1.4, \lambda_2 = -0.7, \vec{v}_1 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

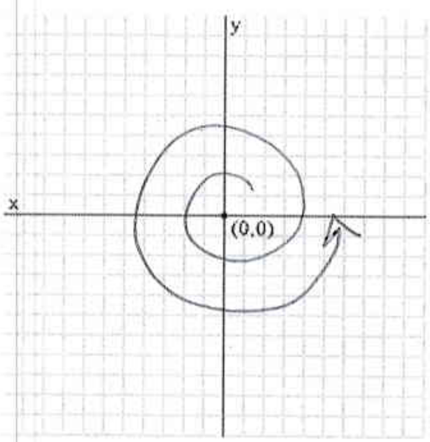
Saddle point



b. $\lambda_1 = -1 + 0.8i, \lambda_2 = -1 - 0.8i$.

repels $|\lambda| > 1$

$\sqrt{1^2 + .8^2} > 1$



5. For the vectors $\vec{u} = \begin{bmatrix} 1 \\ 6 \\ -4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$, find the following: (4 points each)

a. $||\vec{u}||$

$\sqrt{1 + 36 + 16} = \sqrt{53}$

b. A unit vector in the direction of $\vec{u}$.

$\hat{u} = \begin{bmatrix} 1/\sqrt{53} \\ 6/\sqrt{53} \\ -4/\sqrt{53} \end{bmatrix}$

6. Find the determinant of the matrix $\begin{bmatrix} 1 & -2 & 6 \\ 7 & 1 & -1 \\ 2 & 5 & 1 \end{bmatrix}$ by any method. (10 points)

$$1\begin{vmatrix} 1 & -1 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} 7 & -1 \\ 2 & 1 \end{vmatrix} + 6\begin{vmatrix} 7 & 1 \\ 2 & 5 \end{vmatrix} =$$

$$1(1+5) + 2(7+2) + 6(35-2) =$$

$$6 + 18 + 6(33) = 222$$

$$\begin{array}{r} 33 \\ 6 \\ \hline 198 \end{array}$$

7. Given that A and B are 3x3 matrices with det A = -3 and det B = 4, find the following. (4 points each)

a) det AB -12

d) det B^T 4

b) det A^{-1} $-\frac{1}{3}$

e) det 2A $2^3 \cdot (-3) = -24$

c) det $(-AB^3)$ $(-1)^3(-3)4^3 =$
$3 \cdot 64 = 192$

8. Suppose matrix A is a 7×8 matrix with 7 pivot columns. Determine the following. (12 points)

dim Col A = 7

dim Nul A = 1

dim Row A = 7

If Col A is a subspace of $\mathbb{R}^m$, then n = 7

Rank A = 7

If Nul A is a subspace of $\mathbb{R}^n$, then m = 8

9. Determine if the following sets are subspaces. Be sure to check all the necessary conditions or find a counterexample. (6 points each)

a. $W = \left\{ \begin{bmatrix} a+b \\ b+3c \end{bmatrix}, a+b+c=0 \right\}$.

$a=b=c=0$ $\begin{bmatrix} 0+0 \\ 0+3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in space ✓

$d+e+f=0$

$\begin{bmatrix} a+b \\ b+3c \end{bmatrix} + \begin{bmatrix} d+e \\ e+3f \end{bmatrix} = \begin{bmatrix} a+d+b+e \\ b+e+3(c+f) \end{bmatrix}$ $a+d+b+e+c+f=0$

since $(a+b+c)+(d+e+f)=0$ ✓

Sum in set.

$k\begin{bmatrix} a+b \\ b+3c \end{bmatrix} = \begin{bmatrix} ka+kb \\ kb+3kc \end{bmatrix} \Rightarrow$ $ka+kb+kc=0$

$\Rightarrow k(a+b+c)=0$ true

this is a subspace

b. Polynomials of the form $p(t) = a + 2t + bt^2$ as a subspace of P_2.

let $a=b=0 \Rightarrow 0+2t+0t^2 \Rightarrow 2t$ $\vec{0}$ not in the space

this is not a subspace

10. Determine if each statement is True or False. For each of the questions, assume that A is $n \times n$. (3 points each)

a. T F If λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.

b. T F If U has columns that form an orthonormal basis for the column space of U, then $UU^T = I$.

c. T F If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ is also an eigenvector of e^A.

d. T F If the columns of A are linearly dependent, a least squares solution must not exist.

e. T F Row operations on a matrix do not change its eigenvalues.

f. T F The closest point to $\vec{y}$ to a subspace W is the orthogonal projection of the vector onto W.

g. T F The dimension of the eigenspace of an $n \times n$ matrix is always n or less.

h. T F If a vector is a linear combination of nonzero vectors from an orthogonal set, then the weights in the linear combination can be computed without matrix operations.

i. T F Two fundamental questions about linear systems is about existence and uniqueness.

j. T F Stochastic matrices, regardless of their size, always have at least one real eigenvector.

k. T F The $span\{\vec{u}, \vec{v}\}$ is the plane containing all the vectors that are linear combinations of $\vec{u}$ and $\vec{v}$.

l. T F In a discrete dynamical system, the magnitude of λ determines whether a complex eigenvalues causes the origin to repel or attract.

m. T F The cross product is one type of inner product.

n. T F The solution to the system $A\vec{x} = \vec{b}$ is of the form $\vec{x} = \vec{p} + t\vec{v}$ where $\vec{v}$ is any solution to the system $A\vec{x} = \vec{0}$.

o. T F Matrices of the form $\begin{bmatrix} a & b \\ b & a+c \end{bmatrix}$ is a subspace of $M_{2\times2}$.

p. T F If two spaces have the same number of basis vectors, then then are isomorphic.

q. T F The nullspace of an $m \times n$ matrix is a subspace of R^m.

r. T F A linear transformation defined by a 4x6 matrix can be one-to-one, but it cannot be onto.

s. T F A vector space has infinite dimensions if there is no finite basis for the space.

t. T F If A has n pivots, then the system $A\vec{x} = \vec{0}$ does not have a unique solution.

u. T F If the dot product of two vectors is zero, then the vectors are orthogonal.

Math 2568, Final Exam, Part II, Summer 2015 Name ____________________

Instructions: Show all work. You **may** use a calculator on this portion of the exam. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question. Provide explanations where requested.

1. Let $W = span\left\{\begin{bmatrix}1\\-2\\-1\\2\end{bmatrix}, \begin{bmatrix}-4\\1\\0\\3\end{bmatrix}\right\}$. Given $\vec{y} = \begin{bmatrix}1\\-2\\4\\5\end{bmatrix}$, decompose this vector into $\vec{y}_{\parallel}$ in W and $\vec{y}_{\perp}$ in $W^{\perp}$.

   (12 points) w_1 w_2

$$\frac{\vec{y}\cdot\vec{w}_1}{\vec{w}_1\cdot\vec{w}_1}\vec{w}_1 = \frac{1+4-4+10}{1+4+1+4}\begin{bmatrix}1\\-2\\-1\\2\end{bmatrix} = \frac{11}{10}\begin{bmatrix}1\\-2\\-1\\2\end{bmatrix} = \frac{1}{10}\begin{bmatrix}11\\-22\\-11\\22\end{bmatrix} = \frac{1}{260}\begin{bmatrix}286\\-572\\-286\\572\end{bmatrix}$$

$$\frac{\vec{y}\cdot\vec{w}_2}{\vec{w}_2\cdot\vec{w}_2}\vec{w}_2 = \frac{-4-2+0+15}{16+1+9}\begin{bmatrix}-4\\1\\0\\3\end{bmatrix} = \frac{7}{26}\begin{bmatrix}-4\\1\\0\\3\end{bmatrix} = \frac{1}{26}\begin{bmatrix}-28\\7\\0\\21\end{bmatrix} = \frac{1}{260}\begin{bmatrix}-280\\70\\0\\210\end{bmatrix}$$

$$\vec{y}_{\parallel} = \frac{1}{260}\begin{bmatrix}286\\-572\\-286\\572\end{bmatrix} + \frac{1}{260}\begin{bmatrix}-280\\70\\0\\210\end{bmatrix} = \frac{1}{260}\begin{bmatrix}6\\-502\\-286\\782\end{bmatrix} = \frac{1}{130}\begin{bmatrix}3\\-251\\-143\\391\end{bmatrix}$$

$$\vec{y}_{\perp} = \overset{\vec{y}}{\begin{bmatrix}1\\-2\\4\\5\end{bmatrix}} - \vec{y}_{\parallel} = \begin{bmatrix}1-3/130\\-2+251/130\\4+143/130\\5-391/130\end{bmatrix} = \begin{bmatrix}127/130\\-9/130\\51/10\\259/130\end{bmatrix}$$

2. Use least squares to find a linear regression equation $\beta_0 + \beta_1 x = y$ for the data shown in the table below. Be sure to write the final regression equation. Round your answers to two decimal places. (12 points)

x	8	23	13	4	17	23
y	12	34	25	5	26	37

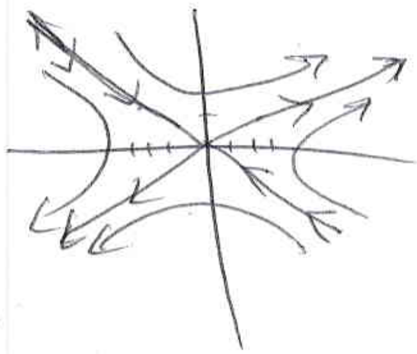
$$A = \begin{bmatrix}1 & 8\\1 & 23\\1 & 13\\1 & 4\\1 & 17\\1 & 23\end{bmatrix} \quad \vec{b} = \begin{bmatrix}12\\34\\25\\5\\26\\37\end{bmatrix}$$

$$\vec{x} = (A^TA)^{-1}A^T\vec{b} \approx \begin{bmatrix}1.24\\1.56\end{bmatrix}$$

$$1.24 + 1.56x = y$$

3. Solve the system of ODEs given by $\vec{x}' = \begin{bmatrix} 2 & 9 \\ 1 & 2 \end{bmatrix}\vec{x}$. Sketch a graph of the eigenvectors and plot some sample trajectories. Is the origin an attractor, a repeller or a saddle point? Give your final solution in exact form with e. (15 points)

$$(2-\lambda)(2-\lambda)-9=$$
$$\lambda^2-4\lambda+4-9=0$$
$$\lambda^2-4\lambda-5=0$$
$$(\lambda-5)(\lambda+1)=0$$
$$\lambda=5,\ \lambda=-1$$



$$\lambda=5 \quad \begin{bmatrix} 2-5 & 9 \\ 1 & 2-5 \end{bmatrix} = \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix} \quad \begin{matrix} x_1-3x_2=0 \\ \Rightarrow x_1=3x_2 \\ x_2=x_2 \end{matrix} \quad \vec{v}_1=\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Saddle point

$$\lambda_2=-1 \quad \begin{bmatrix} 2+1 & 9 \\ 1 & 2+1 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 1 & 3 \end{bmatrix} \quad \begin{matrix} x_1+3x_2=0 \\ \Rightarrow x_1=-3x_2 \\ x_2=x_2 \end{matrix} \quad \vec{v}_2=\begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\vec{x}=c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}e^{5t}+c_2\begin{bmatrix} -3 \\ 1 \end{bmatrix}e^{-t}$$

4. The vectors $\vec{u}=\begin{bmatrix} 2 \\ -5 \\ 1 \\ -1 \end{bmatrix}$, $\vec{v}=\begin{bmatrix} 5 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ are orthogonal to each other. Let $W=span\{\vec{u},\vec{v}\}$. Find an orthogonal basis for $W^{\perp}$. (12 points)

$$\begin{bmatrix} 2 & -5 & 1 & -1 \\ 5 & 2 & 1 & 1 \end{bmatrix} \Rightarrow \text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 7/29 & 3/29 \\ 0 & 1 & -3/29 & 7/29 \end{bmatrix}$$

$$\begin{aligned} x_1 &= -7/29\,x_3 - 3/29\,x_4 \\ x_2 &= 3/29\,x_3 - 7/29\,x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned} \qquad \vec{x}=\begin{bmatrix} -7 \\ 3 \\ 29 \\ 0 \end{bmatrix}t+\begin{bmatrix} -3 \\ -7 \\ 0 \\ 29 \end{bmatrix}s$$

$$W^{\perp}=\text{span}\left\{\begin{bmatrix} -7 \\ 3 \\ 29 \\ 0 \end{bmatrix},\begin{bmatrix} -3 \\ -7 \\ 0 \\ 29 \end{bmatrix}\right\}$$

5. Find the nullspace of the system $\begin{cases} x_1 + 2x_2 - x_3 - 4x_4 + x_5 + 2x_6 = 0 \\ x_1 - 3x_2 \qquad\qquad\qquad + 2x_5 - x_6 = 0 \\ 2x_1 \qquad - x_3 + 2x_4 \qquad\quad + 4x_6 = 0 \end{cases}$. (15 points)

$\Rightarrow \text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 18 & -7 & 8 \\ 0 & 1 & 0 & 6 & -3 & 3 \\ 0 & 0 & 1 & 34 & -14 & 12 \end{bmatrix}$

$x_1 = -18x_4 + 7x_5 - 8x_6$
$x_2 = -6x_4 + 3x_5 - 3x_6$
$x_3 = -34x_4 + 14x_5 - 12x_6$
$x_4 = x_4$
$x_5 = \qquad x_5$
$x_6 = \qquad\qquad x_6$

$\Rightarrow \vec{x} = \begin{bmatrix} -18 \\ -6 \\ -34 \\ 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 7 \\ 3 \\ 14 \\ 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -8 \\ -3 \\ -12 \\ 0 \\ 0 \\ 1 \end{bmatrix} g$

$\text{Nul}\, A = \text{Span}\left\{ \begin{bmatrix} -18 \\ -6 \\ -34 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \\ 14 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ -3 \\ -12 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

6. Determine if the following sets of vectors are linearly independent. Then determine if they form a basis for the specified space. Explain your reasoning. (6 points)

a. $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}, R^4$

reduces to I_4

So it is a basis since it both spans $\mathbb{R}^4$ & is linearly independent

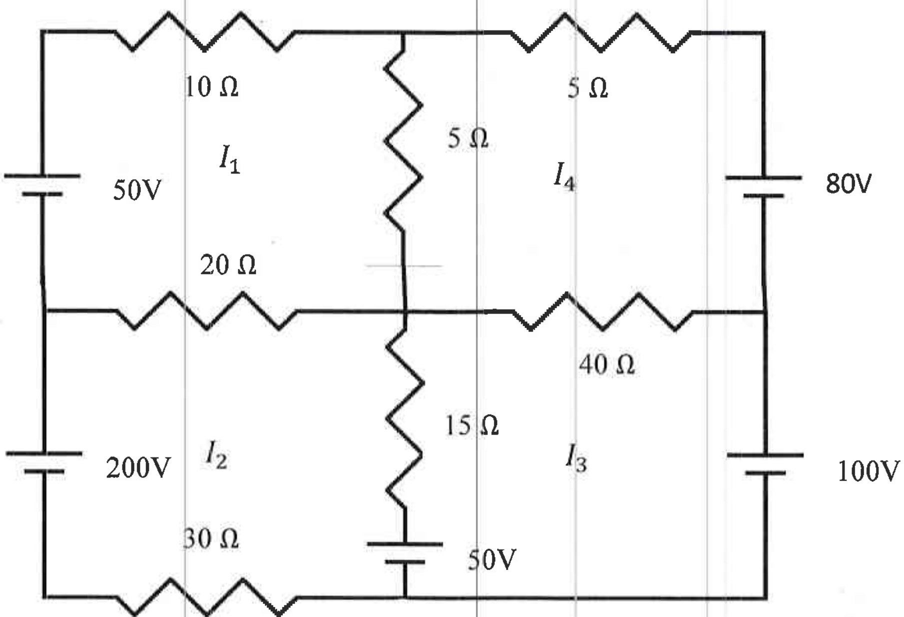
b. $\{1 - t^2, 5 - 3t, t + t^2\}, P_2$

$\begin{bmatrix} 1 & 5 & 0 \\ 0 & -3 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

reduces to I_3

So it is a basis for P_2 since P_2 is isomorphic to $\mathbb{R}^3$

7. Write a matrix to determine the loop currents and use your calculator to solve the system. Round your answers to two decimal places. (15 points)



$$(10+20+5)I_1 - 20I_2 - 5I_4 = -50$$
$$-20I_1 + (20+30+15)I_2 - 15I_3 = -200+50$$
$$-15I_2 + (15+40)I_3 - 40I_4 = -50+100$$
$$-5I_1 \quad -40I_3 + (5+5+40)I_4 = +80$$

$$(35)I_1 - 20I_2 \quad -5I_4 = -50$$
$$-20I_1 + 65I_2 - 15I_3 = -150$$
$$-15I_2 + 65I_3 - 40I_4 = 50$$
$$-5I_1 \quad - \quad -40I_3 + 50I_4 = 80$$

$$\left[\begin{array}{cccc|c} 35 & -20 & 0 & -5 & -50 \\ -20 & 65 & -15 & 0 & -150 \\ 0 & -15 & 65 & -40 & 50 \\ -5 & 0 & -40 & 50 & 80 \end{array}\right] \Rightarrow \quad \vec{I} = \begin{bmatrix} -2.52 \\ -2.63 \\ 1.95 \\ 2.91 \end{bmatrix}$$

8. If a basis for R^3 is $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\}$, and if a vector in the standard basis is $\vec{x} = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}$, find its representation in the basis. [Hint: Is this basis orthogonal?] (12 points)

no.

$$P_B = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 0 & -3 \\ 3 & -2 & 0 \end{bmatrix}$$

$$P_B[\vec{x}]_B = \vec{x}$$

$$P_B^{-1}\vec{x} = [\vec{x}]_B$$

$$P_B^{-1} = \begin{bmatrix} 6/25 & 8/25 & 9/25 \\ 9/25 & 12/25 & 1/25 \\ -2/25 & -11/25 & -3/25 \end{bmatrix}$$

$$P_B^{-1}\begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = [\vec{x}]_B$$