MAT 202, Exam #2, Part I, Summer 2016 Name KEY

Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. When you are finished with this portion of exam, get Part II.

1. Determine if each statement is True or False. (1 point each)

a. T F To find the determinant of a triangular matrix, add the entries on the main diagonal. (multiply)

b. T F Interchanging two rows of a matrix changes the sign of the determinant.

c. T F If one row of a square matrix is a multiple of another row, then the determinant is zero.

d. T F If A is an $n \times n$ matrix and c is a nonzero scalar, then $\det(cA)$ is given by $nc\ \det(A)$. $c^n \det A$

e. T F If the determinant of an $n \times n$ matrix A is nonzero, then $A\vec{x} = \vec{0}$ has only the trivial solution.

f. T F Cramer's Rule can only be used when the determinant of the coefficient matrix is nonzero.

g. T F The vector $-\vec{v}$ is called the additive identity of $\vec{v}$. additive inverse

h. T F The set of all integers with the standard operations is a vector space. scalar multiplication needs inverses, uses real #'s

i. T F Every vector space V contains at least one subspace that is the zero subspace.

j. T F If W is a subspace of R^2, then W must contain the vector $\begin{bmatrix}0\\0\end{bmatrix}$.

k. T F A set $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}, k \geq 2$ is linearly independent if and only if at least one of the vectors $\vec{v}_j$ can be written as a linear combination of the other vectors. this is linearly dependent

l. T F If $\dim(V) = n$ then there exists a set of $n - 1$ vectors in V that will span V. need n vectors to span V

m. T F The nullspace of a matrix A is also called the solution space of A. kernel of A

n. T F The system of linear equations $A\vec{x} = \vec{b}$ is inconsistent if and only if $\vec{b}$ is in the column space of A. consistent

o. T **F** If P is the transition matrix from the standard basis to the basis B, then $P[\vec{x}]_B = \vec{x}$.

P transitions from B to standard basis

p. **T** F The standard operations in R^n are vector addition and scalar multiplication.

q. T **F** The set $W = \left\{\begin{bmatrix} 0 \\ x^2 \\ x^3 \end{bmatrix} : x\ real\right\}$ is a subspace of R^3.

X^2 is a problem for scalar multiplication

r. T **F** Elementary row operations preserve the column space of the matrix A.

preserve row space

s. T **F** Column operations on a matrix do not change the determinant of the matrix.

changes same as row operations

2. Find the determinant of the matrix $\begin{bmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 8 & 1 & 6 \end{bmatrix}$ by the cofactor method. (5 points)

$$3\begin{vmatrix} -5 & 4 \\ 1 & 6 \end{vmatrix} - \cancel{0\begin{vmatrix} 8 & -7 \\ 1 & 6 \end{vmatrix}} + 8\begin{vmatrix} 8 & -7 \\ -5 & 4 \end{vmatrix} =$$

$$3(-30-4) + 8(32-35) = 3(-34) + 8(-3) =$$

$$-102 + (-24) = -126$$

3. Use row-reducing methods to find the determinant of $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 2 & -2 \\ 0 & 3 & 2 & -1 \\ 1 & 2 & 0 & -2 \end{bmatrix}$. (5 points)

$-R_1 + R_4 \to R_4$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 2 & -2 \\ 0 & 3 & 2 & -1 \\ 0 & 0 & 1 & -6 \end{bmatrix} \Rightarrow 1\begin{vmatrix} 1 & 2 & -2 \\ 3 & 2 & -1 \\ 0 & 1 & -6 \end{vmatrix}$$

$-3R_1 + R_2 \to R_2$

$$\begin{vmatrix} 1 & 2 & -2 \\ 0 & -4 & 5 \\ 0 & 1 & -6 \end{vmatrix} \Rightarrow 1\begin{vmatrix} -4 & 5 \\ 1 & -6 \end{vmatrix} = 24-5 = 19$$

4. Use properties of determinants and $A = \begin{bmatrix} 1 & 2 \\ 4 & -2 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}$ to find: (2 points each)

$\det A = -2-8 = -10$

$\det B = 0-6 = -6$

a. $\det(AB)$

$(-10)(-6) = 60$

b. $\det(A^2)$

$(-10)^2 = 100$

c. $\det(B^{-1})$

$\frac{1}{-6}$

d. $\det(3A)$

$3^2(-10) = -90$

e. $\det(A^{-1}B^T)$

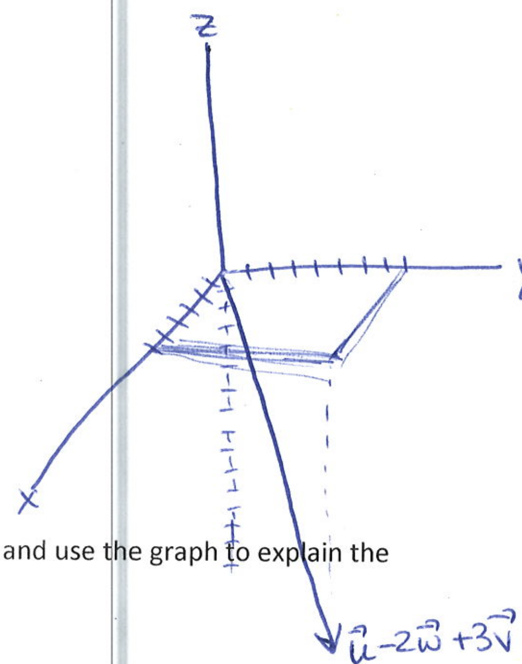
$-\frac{1}{10}(-6) = \frac{6}{10} = \frac{3}{5}$

5. For the vectors $\vec{u} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}, \vec{w} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$, find $\vec{u} - 2\vec{w} + 3\vec{v}$ and sketch the vector that results. (3 points)

$$\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} =$$

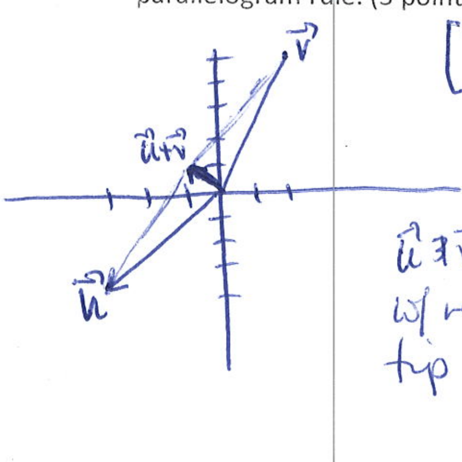
$$\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -8 \\ 6 \\ -2 \end{bmatrix} + \begin{bmatrix} 12 \\ 0 \\ -12 \end{bmatrix} =$$

$$\begin{bmatrix} 6 \\ 8 \\ -15 \end{bmatrix}$$



6. Graph the vectors $\vec{u} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$. Draw $\vec{u} + \vec{v}$ and use the graph to explain the parallelogram rule. (3 points)

$$\begin{bmatrix} -3 \\ -4 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



$\vec{u}$ & $\vec{v}$ form the legs of a parallelogram w/ remaining legs extending $\vec{v}$ from tip of $\vec{u}$ & $\vec{u}$ from tip of $\vec{v}$. The diagonal is $\vec{u}+\vec{v}$.

7. Determine if each of the following sets is a subspace or vector space. If it is a subspace, prove it. If it is not, provide an example of where it fails a property and state which property it fails. (4 points each)

a. $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$ not a subspace

fails scalar multiplication, since scalars must be in $\mathbb{R}$

$\vec{u} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ in H, but $-1\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$ which is not in H.

b. The set of all singular 3×3 matrices.

not a subspace. fails addition

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ in set & $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ in set, but $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

which is nonsingular and so not in set.

c. $V = \{p(t) = at^5 + bt^4 + ct^2 : a, b, c \text{ real}\}$ is a subspace.

i) if $a=b=c=0$ then $p(t)=0$ so the zero vector in set.

ii) $p(t) = at^5 + bt^4 + ct^2$ and $q(t) = dt^5 + et^4 + ft^2$, then $p(t)+q(t) = (a+d)t^5 + (b+e)t^4 + (c+f)t^2$. Since $a+d, b+e, c+f$ real, its in V.

iii) $kp(t) = kat^5 + kbt^4 + kct^2$. Since ka, kb, kc real, its in V.

8. Determine by inspection, if each set of vectors is linearly independent. Explain your reasoning. (2 points each)

a. $\left\{ \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$ independent; only 2 vectors, not multiples of each other

b. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ independent

There is a pivot in every column.

c. $\{x^2 - 1, 2x + 5\}$ independent

2 vectors, not scalar multiples

d. $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}2\\3\\6\end{bmatrix},\begin{bmatrix}4\\4\\4\end{bmatrix}\right\}$ dependent, since $4\vec{v_1} = \vec{v_3}$.

9. Consider the basis for R^3, $B = \left\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}2\\3\\1\end{bmatrix},\begin{bmatrix}-1\\0\\2\end{bmatrix}\right\}$ and $[\vec{x}]_B = \begin{bmatrix}4\\-3\\-2\end{bmatrix}$. Write the vector in the standard basis. (3 points)

$$4\begin{bmatrix}1\\2\\-1\end{bmatrix} + (-3)\begin{bmatrix}2\\3\\1\end{bmatrix} + (-2)\begin{bmatrix}-1\\0\\2\end{bmatrix} =$$

$$\begin{bmatrix}4\\8\\-4\end{bmatrix} + \begin{bmatrix}-6\\-9\\-3\end{bmatrix} + \begin{bmatrix}2\\0\\-4\end{bmatrix} = \begin{bmatrix}0\\-1\\-11\end{bmatrix}$$

MAT 202, Exam #2, Part II, Summer 2016 Name ______________________

Instructions: Show all work. You **may** use a calculator on this portion of the exam. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question. Provide explanations where requested.

1. Use Cramer's Rule to solve the system $\begin{cases} 4x_1 - x_2 + x_3 = -5 \\ 2x_1 + 2x_2 + 3x_3 = 10 \\ 5x_1 - 2x_2 + 6x_3 = 1 \end{cases}$. Show all the required matrices. You may use your calculator to find the required determinants. State your final solution as a vector. (5 points)

$A = \begin{bmatrix} 4 & -1 & 1 \\ 2 & 2 & 3 \\ 5 & -2 & 6 \end{bmatrix}$ $\det A = 55$ $A_1 = \begin{bmatrix} -5 & -1 & 1 \\ 10 & 2 & 3 \\ 1 & -2 & 6 \end{bmatrix}$ $\det A_1 = -55$

$A_2 = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 10 & 3 \\ 5 & 1 & 6 \end{bmatrix}$ $\det A_2 = 165$ $A_3 = \begin{bmatrix} 4 & -1 & -5 \\ 2 & 2 & 10 \\ 5 & -2 & 1 \end{bmatrix}$ $\det A_3 = 110$

$x_1 = \frac{-55}{55} = -1$ $x_2 = \frac{165}{55} = 3$ $x_3 = \frac{110}{55} = 2$

$$\vec{x} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$

2. Find the area of the triangle given by the vertices $(-1,2), (2,2), (-2,4)$. (3 points)

$$\frac{1}{2}\left|\begin{vmatrix} -1 & 2 & 1 \\ 2 & 2 & 1 \\ -2 & 4 & 1 \end{vmatrix}\right| = \frac{1}{2}(6) = 3$$

3. Find the volume of the parallelepiped whose corner is defined by the vectors $\begin{bmatrix}1\\-2\\3\end{bmatrix}, \begin{bmatrix}4\\0\\1\end{bmatrix}, \begin{bmatrix}2\\5\\-3\end{bmatrix}$.

(3 points)

$$\begin{vmatrix}1 & 4 & 2\\-2 & 0 & 5\\3 & 1 & -3\end{vmatrix} = 27$$

4. Write $\vec{v} = \begin{bmatrix}2\\5\\-4\\0\end{bmatrix}$ as a linear combination of $\vec{u}_1 = \begin{bmatrix}1\\3\\2\\1\end{bmatrix}, \vec{u}_2 = \begin{bmatrix}2\\-2\\-5\\4\end{bmatrix}, \vec{u}_3 = \begin{bmatrix}2\\-1\\3\\6\end{bmatrix}$. (3 points)

$$\left[\begin{array}{ccc|c}1 & 2 & 2 & 2\\3 & -2 & -1 & 5\\2 & -5 & 3 & -4\\1 & 4 & 6 & 0\end{array}\right] \text{ rref} \Rightarrow \left[\begin{array}{ccc|c}1 & 0 & 0 & 2\\0 & 1 & 0 & 1\\0 & 0 & 1 & -1\\0 & 0 & 0 & 0\end{array}\right]$$

$2\vec{u} + \vec{u}_2 + (-1)\vec{u}_3$

5. Find a spanning set for the space spanned by the vectors $\left\{\begin{bmatrix}1\\0\\3\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\4\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\10\\4\end{bmatrix}, \begin{bmatrix}5\\2\\0\\6\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}$. (4 points)

put in 4x6 matrix

$$\text{rref} \Rightarrow \begin{bmatrix}1 & 0 & 2 & 0 & 0 & -1/4\\0 & 1 & 1 & 0 & 0 & 1/4\\0 & 0 & 0 & 1 & 0 & 1/4\\0 & 0 & 0 & 0 & 1 & 3/4\end{bmatrix}$$

$\uparrow$ no pivot $\qquad \uparrow$ no pivot $\leftarrow$ remove

Spanning set $\left\{\begin{bmatrix}1\\0\\3\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\4\\0\end{bmatrix}, \begin{bmatrix}5\\2\\0\\6\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\}$

6. Determine whether $S = \{t^3 - 1, 2t^2, t + 3, 5 + 2t + 2t^2 + t^3\}$ is a basis for P_3. (3 points)

1 2 3 4

$$\begin{bmatrix} -1 & 0 & 3 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ rref} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

yes independent & Spans space so it is a basis for P_3.

7. Consider the matrix $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$. Find a basis for:

a. The nullspace of A (3 points)

$$\text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_3 - x_5$
$x_2 = 2x_3 - 3x_5$
$x_3 = x_3$
$x_4 = 5x_5$
$x_5 = x_5$

$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

b. The column space of A (3 points)

$$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

c. The rank of A (2 points)

3

8. Find the change of basis matrix to transition from C to B for $B = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ -5 \\ 11 \end{bmatrix} \right\}$, $C = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Use it to convert $[\vec{x}]_C = \begin{bmatrix} 0 \\ -2 \\ 4 \\ 5 \end{bmatrix}$ to $[\vec{x}]_B$. (4 points)

$$P_C[\vec{x}]_C = P_B[\vec{x}]_B \Rightarrow \underbrace{P_B^{-1}P_C}_{\underset{B\leftarrow C}{P}}[\vec{x}]_C = [\vec{x}]_B$$

$$P_B^{-1}P_C = \underset{B\leftarrow C}{P} = \begin{bmatrix} -4 & 27 & 13 & -33 \\ -3 & 9 & 5 & -14 \\ -1 & 45 & 17 & -38 \\ 1 & -17 & -7 & 16 \end{bmatrix} \qquad \underset{B\leftarrow C}{P}\begin{bmatrix} 0 \\ -2 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -167 \\ -68 \\ -212 \\ 86 \end{bmatrix} = [\vec{x}]_B$$

9. If A and P are $n \times n$ and P is invertible, does $P^{-1}AP = A$? Illustrate your conclusion with an example. What can you say about $\det(P^{-1}AP)$ and $\det(A)$? (3 points)

$$P = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \quad P^{-1}AP = \begin{bmatrix} 7/2 & 5/2 \\ -3/2 & -1/2 \end{bmatrix} \therefore \text{no}$$

$P^{-1}AP \neq A$ in general since matrix multiplication is not commutative. However $\det A = \det(P^{-1}AP)$ since $= \det P^{-1} \det A \det P = \det P^{-1} \det P \det A = \det(P^{-1}P)\det A = \det I \det A = \det A$.

10. Prove that any set of vectors containing the zero vector is linearly dependent. (3 points)

Consider the set $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k, \vec{0}\}$

Since $\vec{0}\vec{u}_i$ is a linear combination of $\vec{u}_i$,

$\vec{0} = 0\vec{u}_i$ for any $\vec{u}_i$ in the set. $\therefore$ not independent.

11. Find a basis for the vector space of all 3×3 symmetric matrices. What is the dimension of this space? (4 points)

Since $a_{ij} = a_{ji}$

we have $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ so a basis is: (standard)

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

the dimension of This space is 6.