202 Homework #3 key

1a. $\det\left(\begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}\right) = (-1(2) - 4(3)) = -2 - 12 = -14$

b. $\begin{vmatrix} 4 & 3 & 0 \\ 6 & 5 & 2 \\ 9 & 7 & 3 \end{vmatrix} = 4\begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix} - 3\begin{vmatrix} 6 & 2 \\ 9 & 3 \end{vmatrix} + \cancel{0\begin{vmatrix} 6 & 5 \\ 9 & 7 \end{vmatrix}} =$

$4(15-14) - 3(\cancel{18-18}) = 4(1) = 4$

c. $\begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix} = 2\begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 8 \end{vmatrix} = 2(5)\begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix} = 10(7-6) = 10$

d. $\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix} = 1\begin{vmatrix} 6 & 3 & 2 & 4 \\ 9 & 0 & -4 & 1 \\ 3 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 \end{vmatrix} = 3\begin{vmatrix} 3 & 2 & 4 \\ 0 & -4 & 1 \\ 2 & 3 & 2 \end{vmatrix} = 3\left[3\begin{vmatrix} -4 & 1 \\ 3 & 2 \end{vmatrix} + \right.$

$\left. \cancel{-0\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}} + 2\begin{vmatrix} 2 & 4 \\ -4 & 1 \end{vmatrix}\right] = 3\left[3(-8-3) + 2(2+16)\right] =$

$3[-33 + 36] = 3(3) = 9$

e. $\begin{vmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = k(1)(1) = k$

f. $\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1$

2a. $\begin{bmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{bmatrix}$ $-3R_1 + R_2 \to R_2$, $-2R_1 + R_3 \to R_3$, no change $\begin{bmatrix} 1 & 5 & -3 \\ 0 & -18 & 12 \\ 0 & 3 & -1 \end{bmatrix}$ $R_2 \leftrightarrow R_3$ (-1) $\begin{bmatrix} 1 & 5 & -3 \\ 0 & 3 & -1 \\ 0 & -18 & 12 \end{bmatrix}$

$6R_2 + R_3 \to R_3$ $\begin{bmatrix} 1 & 5 & -3 \\ 0 & 3 & -1 \\ 0 & 0 & 6 \end{bmatrix}$ $\det(A) = (-1)(1)(3)(6) = -18$

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2b. $\begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{bmatrix}$ $R_1+R_3 \to R_3$, $-3R_1+R_4 \to R_4$, no change $\begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 1 & 5 & 5 \\ 0 & 2 & 7 & 3 \end{bmatrix}$

$-R_2+R_3 \to R_3$, $-2R_2+R_4 \to R_4$, no change $\begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & -5 \end{bmatrix}$ $R_3 \leftrightarrow R_4$ (-1) $\begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\det(A) = (-1)(1)(1)(-3)(1) = 3$

2c. $\begin{bmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & 7 \end{bmatrix}$ $2R_1+R_3 \to R_3$, $-3R_1+R_4 \to R_4$, $-3R_1+R_5 \to R_5$, no change $\begin{bmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & -3 & 1 & 3 & 7 \\ 0 & -2 & 0 & 8 & -1 \\ 0 & -4 & 8 & 2 & 13 \end{bmatrix}$

$\frac{3}{2}R_2+R_3 \to R_3$, $R_2+R_4 \to R_4$, $2R_2+R_5 \to R_5$, no change $\begin{bmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & -5 & 3/2 & -2 \\ 0 & 0 & -4 & 7 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $-\frac{4}{5}R_3+R_4 \to R_4$

$0 \; 0 \; 4 \; -6/5 \; 8/5$

no change

$\begin{bmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & -5 & 3/2 & -2 \\ 0 & 0 & 0 & 29/5 & -27/5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $\det(A) = 1(2)(-5)(29/5)(1) = -58$

d. $\begin{bmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{bmatrix}$ $R_1+R_4 \to R_4$, $-R_1+R_3 \to R_3$, no change $\begin{bmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ 0 & 6 & -3 & 12 \\ 0 & -6 & 1 & 0 \end{bmatrix}$ $R_1 \leftrightarrow R_2$ (-1)

$\begin{bmatrix} 1 & 3 & 0 & -3 \\ -3 & -2 & 1 & -4 \\ 0 & 6 & -3 & 12 \\ 0 & -6 & 1 & 0 \end{bmatrix}$ $3R_1+R_2 \to R_2$, no change $\begin{bmatrix} 1 & 3 & 0 & -3 \\ 0 & 7 & 1 & -13 \\ 0 & 6 & -3 & 12 \\ 0 & -6 & 1 & 0 \end{bmatrix}$ $R_2+R_4 \to R_4$

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2d cont'd

$$\begin{bmatrix} 1 & 3 & 0 & -3 \\ 0 & 7 & 1 & -13 \\ 0 & 6 & -3 & 12 \\ 0 & 1 & 2 & -13 \end{bmatrix} \quad \underset{(-1)}{R_2 \leftrightarrow R_4} \quad \begin{bmatrix} 1 & 3 & 0 & -3 \\ 0 & 1 & 2 & -13 \\ 0 & 6 & -3 & 12 \\ 0 & 7 & 1 & -13 \end{bmatrix}$$

$-6R_2 + R_3 \to R_3$
$-7R_2 + R_4 \to R_4$
no change

$$\begin{bmatrix} 1 & 3 & 0 & -3 \\ 0 & 1 & 2 & -13 \\ 0 & 0 & -15 & 90 \\ 0 & 0 & -13 & 78 \end{bmatrix} \quad \underset{(-15)}{-\tfrac{1}{15}R_3 \to R_3} \quad \begin{bmatrix} 1 & 3 & 0 & -3 \\ 0 & 1 & 2 & -13 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & -13 & 78 \end{bmatrix}$$

to change back

$13R_3 + R_4 \to R_4$
no change

$$\begin{bmatrix} 1 & 3 & 0 & -3 \\ 0 & 1 & 2 & -13 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \det A = (-1)(-1)(-15)(1)(1)(1)(0) = 0$$

3a. operation $3R_2 \to R_2$ $\quad \det = 7(3) = 21$

b. operations $\underset{(-1)}{R_2 \leftrightarrow R_3}$, $\underset{(2)}{R_1 + 2R_3 \to R_3}$ $\quad \det = 7(2)(-1) = -14$

4. matrices in a, b, c are invertible; d is not

5a. Since $AA^{-1} = I$, using properties that $\det(AB) = \det A \cdot \det B$ and taking the determenant of both sides we get $\det(AA^{-1}) = \det I$, and so $\det A \cdot \det A^{-1} = \det I$. The determinant of the identity is one, since I is diagonal and its determinant is the product of its diagonal entries. Thus $\det A \cdot \det A^{-1} = 1$. If A is invertible, then $\det A \neq 0$ so we can divide. $\det A^{-1} = \dfrac{1}{\det A}$.

b. If A & B are square and both are $n \times n$, then AB is defined and so is BA. $\det(AB) = \det A \cdot \det B$ by products property of determinants. Since $\det A$ and $\det B$ are real number, by commutativity $\det A \cdot \det B = \det B \cdot \det A$ and by product property of determinants $\det B \cdot \det A = \det(BA)$, thus $\det(AB) = \det(BA)$.

5c. if A, P are $n \times n$ and P invertible then PAP^{-1} is defined.
$\det(PAP^{-1}) = \det P \cdot \det(AP^{-1}) = \det P \cdot \det A \cdot \det P^{-1}$ by successive application of the product property. Since these are real numbers, by commutativity $\Rightarrow \det P \cdot \det P^{-1} \det A$ and by product property this is $\det(PP^{-1}) \det A = \det(I) \det A$ by definition of the inverse. Since $\det I = 1$, this $= \det A$.
Thus $\det(PAP^{-1}) = \det(A)$.

d. if $\det(A^4) = 0$ then by the product property $\det(A^4) = \det(A \cdot A \cdot A \cdot A) = \det A \cdot \det A \cdot \det A \cdot \det A = (\det A)^4 = 0$
eliminating the exponent $((\det A)^4)^{1/4} = 0^{1/4} \Rightarrow \det A = 0$.
any matrix whose determinant is zero is singular and not invertible.

6 i. $\det A = -1$, $\det B = 2$
$\det AB = \det A \cdot \det B = (-1)(2) = -2$

ii. $\det(B^5) = (\det B)^5 = (2)^5 = 32$

iii. $\det(2A)$ since 2 multiplies each row, this is n copies of 2, so
$= 2^n \det A = (-1)2^n$

iv. $\det(A^TA) = \det A^T \cdot \det A$ since $\det A^T = \det A$
$= (\det A) \cdot \det A = (-1)(-1) = 1$

v. $\det(B^{-1}AB) = \det(B^{-1}) \det A \det B = (1/2)(-1)(2) = -1$

7a. $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ $A_1 = \begin{bmatrix} -10 & -1 \\ -1 & 2 \end{bmatrix}$ $A_2 = \begin{bmatrix} 2 & -10 \\ 3 & -1 \end{bmatrix}$

$\det A = 4 + 3 = 7$ $\det A_1 = -20 - 1 = -21$ $\det A_2 = -2 + 30 = 28$

$x_1 = \frac{-21}{7} = -3$ $x_2 = \frac{28}{7} = 4$ Solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

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7b. $A = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 2 & 5 \\ 8 & -5 & -2 \end{bmatrix}$ $A_1 = \begin{bmatrix} -2 & -2 & 3 \\ 16 & 2 & 5 \\ 4 & -5 & -2 \end{bmatrix}$ $A_2 = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 16 & 5 \\ 8 & 4 & -2 \end{bmatrix}$

$A_3 = \begin{bmatrix} 4 & -2 & -2 \\ 2 & 2 & 16 \\ 8 & -5 & 4 \end{bmatrix}$ $\det A = -82$ $\det A_2 = -656$

$\det A_1 = -410$ $\det A_3 = 164$

$x_1 = \dfrac{-410}{-82} = 5$ $x_2 = \dfrac{-656}{-82} = 8$ $x_3 = \dfrac{164}{-82} = -2$

Solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ -2 \end{bmatrix}$

c. $A = \begin{bmatrix} -1 & -1 & 0 & 1 \\ 3 & 5 & 5 & 0 \\ 0 & 0 & 2 & 1 \\ -2 & -3 & -3 & 0 \end{bmatrix}$ $A_1 = \begin{bmatrix} -8 & -1 & 0 & 1 \\ 24 & 5 & 5 & 0 \\ -6 & 0 & 2 & 1 \\ 15 & -3 & -3 & 0 \end{bmatrix}$ $A_2 = \begin{bmatrix} -1 & -8 & 0 & 1 \\ 3 & 24 & 5 & 0 \\ 0 & -6 & 2 & 1 \\ -2 & 15 & -3 & 0 \end{bmatrix}$

$A_3 = \begin{bmatrix} -1 & -1 & -8 & 1 \\ 3 & 5 & 24 & 0 \\ 0 & 0 & -6 & 1 \\ -2 & -3 & 15 & 0 \end{bmatrix}$ $A_4 = \begin{bmatrix} -1 & -1 & 0 & -8 \\ 3 & 5 & 5 & 24 \\ 0 & 0 & 2 & -6 \\ -2 & -3 & -3 & 15 \end{bmatrix}$

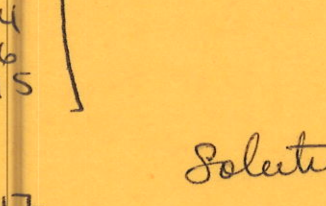
$\det A = 1$
$\det A_1 = -147$
$\det A_2 = 37$
$\det A_3 = 56$
$\det A_4 = -118$

$x_1 = \dfrac{-147}{1} = -147$
$x_2 = \dfrac{37}{1} = 37$
$x_3 = \dfrac{56}{1} = 56$
$x_4 = \dfrac{-118}{1} = -118$

Solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -147 \\ 37 \\ 56 \\ -118 \end{bmatrix}$

8. a. $\begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}$ $|\det A| = |2+12| = 14$

b. $\vec{u} = \begin{bmatrix} -1-1 \\ 2-0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$



$\vec{v} = \begin{bmatrix} 6-1 \\ 2-0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$\begin{bmatrix} -2 & 5 \\ 2 & 2 \end{bmatrix}$ $|\det A| = |-4-10| = -14$

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9. $V = \left| \det \begin{bmatrix} -2 & 1 & 6 \\ 3 & 8 & 1 \\ 7 & 4 & -1 \end{bmatrix} \right| = |-230| = 230$

10. $V = \frac{1}{6} \left| \det \begin{bmatrix} 1 & -2 & -3 \\ -3 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix} \right|$

$\vec{u} = \begin{bmatrix} 4-3 \\ -4+1 \\ 4-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 1-3 \\ 1+1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$

$\vec{w} = \begin{bmatrix} 0-3 \\ 0+1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

$\frac{1}{6}|12| = 2$

11.
a. false. it may depending on whether the row being replaced was changed by something other than addition of another row.

b. true

c. false. Only if the matrix is triangular or diagonal

d. false. any linear combination of other rows will make the determinant zero.

e. false. $\det A = \det(A^T)$.