202 Homework #5 key

1. a. true
b. false. $A\vec{x}=\vec{b}$ is a matrix equation. $\vec{a}_1x_1+\vec{a}_2x_2+\vec{a}_3x_3+\cdots+\vec{a}_nx_n=\vec{b}$ is a vector equation.
c. false. pivot in every column (but the last), or rows of zeros are consistent. If the augmented matrix has a pivot in last row last column it is inconsistent.
d. false. There may be $\vec{b}$ in $\mathbb{R}^m$ (if $m>n$) where the system is inconsistent.
e. false. trivial solution has no free variables.
f. false. line through $\vec{p}$, parallel to $\vec{v}$.
g. true
h. true. as long as the vector is not $\vec{0}$
i. false. the set also needs to be independent.
j. true
k. false. as small as possible
l. false.
m. false. must also span H.
n. true.
o. true
p. false. its in $\mathbb{R}^n$
q. true
r. true
s. false. set of all $\vec{b}$'s so that $A\vec{x}=\vec{b}$ has a solution.
t. false
u. false. pivot columns must be taken from A.
v. true
w. false. P_3 isomorphic to $\mathbb{R}^4$.
x. true
y. true
z. true
aa. false. The # of columns of A.
bb. true

202 Homework #5 key

1cc. true

dd. true

ee. true

ff. false. only if the matrix is $n \times n$.

gg. false. system could be dependent.

hh. false. $P_B [\vec{x}]_B = \vec{x}$

ii. false. They are C-coordinate vectors of the vectors in B.

jj. true

2. a. $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

c. 4 pivots, one for each column. It can't have more than that.

3. $\begin{bmatrix} 2 \\ 6 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ 6 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 10 \\ 6 \\ -6 \\ 2 \end{bmatrix}$ etc. answers will vary

4. a. rref $\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ they do not span $\mathbb{R}^4$. only 3 pivots.

b. rref $\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ they do span $\mathbb{R}^4$ as there are 4 pivots.

5. a. independent; one pivot in each column

b. reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ There is a pivot in every column, so independent

c. 2 vectors, not multiples, so independent.

d. 4 vectors, more than rows of matrix, dependent.

6. a. no pivot in column 3 - remove vector; basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 10 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -6 \\ 9 \end{bmatrix} \right\}$

b. no pivot in column 4 - remove; basis is $\left\{ \begin{bmatrix} -3 \\ 2 \\ 6 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -9 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -4 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$

202 Homework #5 key

7a. $\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix} \Rightarrow \text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 3 & -2 \end{bmatrix}$

$x_1 = 2x_3 - 4x_4$
$x_2 = -3x_3 + 2x_4$
$x_3 = x_3$
$x_4 = x_4$

$\Rightarrow \vec{x} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix} x_4$

$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

b. $\begin{bmatrix} 1 & 3 & -4 & -3 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 5 & -6 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 = -5x_3 + 6x_4 - x_5$
$x_2 = 3x_3 - x_4$
$x_3 = x_3$
$x_4 = x_4$
$x_5 = x_5$

$\vec{x} = \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_5$

$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

8a. 2 pivots $\Rightarrow$ Rank $A = 2$ by definition
dim Row $A = 2$ by definition
dim Nul $A = 5 - 2 = 3$ since Rank A + dim Nul $A = n$ # of columns

b. dim Nul $A = 8 - 4 = 4$; Col A is isomorphic to $\mathbb{R}^4$, but since its made up of vectors in $\mathbb{R}^6$, it is not equal to $\mathbb{R}^4$.

c. n - dim Nul $A = 7 - 5 = 2$ dim Col $A = 2$ since there are two pivots.

d. A can have a maximum 4 pivots, so dim Row A can only be 4 or smaller.

e. The smallest possible dimension for Nul A is zero since there are five columns and each could have a pivot.

f. matrix is 6x5. the system will not always be consistent. Nul A has one dimension (one non-zero solution & its multiples), this means there are only 4 pivots. if Col A is only dimension 4, it can't span $\mathbb{R}^6$, so there will be some values of $\vec{b}$ that can't be reached.

202 Homework #5 Key

9a. $\begin{bmatrix}1\\2\\0\\-3\end{bmatrix}p+\begin{bmatrix}-2\\0\\-2\\0\end{bmatrix}q+\begin{bmatrix}0\\-5\\2\\6\end{bmatrix}r \quad \text{rref} \Rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$ these are independent & span space

basis $= \left\{\begin{bmatrix}1\\2\\0\\-3\end{bmatrix},\begin{bmatrix}-2\\0\\-2\\0\end{bmatrix},\begin{bmatrix}0\\-5\\2\\6\end{bmatrix}\right\}$

b. rref $\Rightarrow \begin{bmatrix}1&0&0&-1\\0&1&0&0\\0&0&1&1\end{bmatrix}$ basis $= \left\{\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}-3\\-6\\0\end{bmatrix},\begin{bmatrix}-2\\3\\5\end{bmatrix}\right\}$

c. rref $\Rightarrow \begin{bmatrix}1&2&-4&0&0&23&0\\0&0&0&1&0&-3&0\\0&0&0&0&1&4&0\\0&0&0&0&0&0&1\end{bmatrix}$ Col A $= \left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}3\\1\\0\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\7\\-2\\1\end{bmatrix}\right\}$

$x_1 = -2x_2 + 4x_3 - 23x_6$
$x_2 = x_2$
$x_3 = x_3$
$x_4 = 3x_6$
$x_5 = -4x_6$
$x_6 = x_6$
$x_7 = 0$

$\vec{x} = \begin{bmatrix}-2\\1\\0\\0\\0\\0\\0\end{bmatrix}x_2 + \begin{bmatrix}4\\0\\1\\0\\0\\0\\0\end{bmatrix}x_3 + \begin{bmatrix}-23\\0\\0\\3\\-4\\1\\0\end{bmatrix}x_6$

Nul A $= \left\{\begin{bmatrix}-2\\1\\0\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}4\\0\\1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}-23\\0\\0\\3\\-4\\1\\0\end{bmatrix}\right\}$

d. $\begin{bmatrix}1&1&2&6\\0&-1&-4&-18\\0&0&1&9\\0&0&0&-1\end{bmatrix}$ this is a basis; pivot in every row & column
$\{1, 1-t, 2-4t+t^2, 6-18t+9t^2-t^3\}$

10. Col A $= \left\{\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\-1\\-2\\-2\end{bmatrix},\begin{bmatrix}-2\\-3\\0\\2\\1\end{bmatrix},\begin{bmatrix}1\\-2\\1\\-3\\2\end{bmatrix},\begin{bmatrix}-2\\-3\\6\\0\\-1\end{bmatrix}\right\}$ Row A $= \left\{\begin{bmatrix}1\\1\\-2\\-4\\1\\-2\end{bmatrix},\begin{bmatrix}0\\1\\-1\\0\\-3\\-1\end{bmatrix},\begin{bmatrix}0\\0\\1\\1\\-13\\-1\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\1\\-1\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}\right\}$

rank A = 5 dim Nul A = 1 $\quad$ rref $\overset{B}{\Rightarrow} \begin{bmatrix}1&0&0&-3&0&0\\0&1&0&1&0&0\\0&0&1&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}$

$x_1 = 3x_4$
$x_2 = -x_4$
$x_3 = -x_4$
$x_4 = x_4$
$x_5 = 0$
$x_6 = 0$

$\vec{x} = \begin{bmatrix}3\\-1\\-1\\1\\0\\0\end{bmatrix}x_4$

Nul A $= \left\{\begin{bmatrix}3\\-1\\-1\\1\\0\\0\end{bmatrix}\right\}$

202 Homework #5 Key

11a. $P_B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 4 & 3 & 1 \end{bmatrix}$ $\quad P_C = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 4 \\ 2 & 1 & 5 \end{bmatrix}$

$$P_C [\vec{x}]_C = P_B [\vec{x}]_B$$

$$\underbrace{P_B^{-1} P_C}_{P_{B \leftarrow C}} [\vec{x}]_C = [\vec{x}]_B$$

$$P_B^{-1} = \begin{bmatrix} -5/11 & 4/11 & 2/11 \\ 6/11 & -7/11 & 2/11 \\ 2/11 & 5/11 & -3/11 \end{bmatrix}$$

$$P_B^{-1} P_C = P_{B \leftarrow C} = \begin{bmatrix} -5/11 & 6/11 & 1 \\ 17/11 & -5/11 & 0 \\ -9/11 & 2/11 & 1 \end{bmatrix}$$

$$[\vec{x}]_B = P_{B \leftarrow C} \begin{bmatrix} -4 \\ 10 \\ 11 \end{bmatrix} = \begin{bmatrix} 201/11 \\ -118/11 \\ 177/11 \end{bmatrix}$$

b. $P_B = \begin{bmatrix} 1 & 5 & 2 & 0 \\ -2 & 1 & 2 & 0 \\ 2 & 2 & 1 & 3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ $\quad P_C = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 3 & 1 & 0 & 4 \end{bmatrix}$

$$P_C [\vec{x}]_C = P_B [\vec{x}]_B$$

$$[\vec{x}]_C = \underbrace{P_C^{-1} P_B}_{P_{C \leftarrow B}} [\vec{x}]_B$$

$$P_C^{-1} = \begin{bmatrix} -4/17 & 11/17 & 8/17 & 1/17 \\ 8/17 & -5/17 & -16/17 & -2/17 \\ 8/17 & -5/17 & 1/17 & -2/17 \\ 1/17 & -7/17 & -2/17 & 4/17 \end{bmatrix}$$

$$P_C^{-1} P_B = P_{C \leftarrow B} = \begin{bmatrix} -11/17 & 8/17 & 22/17 & 25/17 \\ -12/17 & 1/17 & -10/17 & -50/17 \\ 24/17 & 35/17 & 7/17 & 1/17 \\ 7/17 & -2/17 & -14/17 & -2/17 \end{bmatrix}$$

$$[\vec{x}]_C = P_{C \leftarrow B} \begin{bmatrix} 5 \\ 2 \\ -3 \\ 10 \end{bmatrix} = \begin{bmatrix} 145/17 \\ -528/17 \\ 169/17 \\ 53/17 \end{bmatrix}$$