202 Homework #6 Key

1\. a. $\vec{u}\cdot\vec{v}$

i. $(-1)(4)+2(6) = -4+12 = 8$

ii. $(12)(2)+3(-3)+(-5)(3) = 24-9-15 = 0$

iii. $(3)(-4)+2(1)+(-5)(-2)+0(6) = -12+2+10+0 = 0$

b. $\|\vec{u}\|$ and $\|\vec{v}\|$

i. $\|\vec{u}\| = \sqrt{1^2+2^2} = \sqrt{5}$ $\quad \|\vec{v}\| = \sqrt{16+36} = \sqrt{52} = 2\sqrt{13}$

ii. $\|\vec{u}\| = \sqrt{144+9+25} = \sqrt{178}$ $\quad \|\vec{v}\| = \sqrt{4+9+9} = \sqrt{22}$

iii. $\|\vec{u}\| = \sqrt{9+4+25} = \sqrt{38}$ $\quad \|\vec{v}\| = \sqrt{16+1+4+36} = \sqrt{57}$

c. $\dfrac{\vec{u}}{\|u\|}$ and $\dfrac{\vec{v}}{\|v\|}$

i. $\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \hat{u}$, $\hat{v} = \begin{bmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}$

ii. $\hat{u} = \begin{bmatrix} 12/\sqrt{178} \\ 3/\sqrt{178} \\ -5/\sqrt{178} \end{bmatrix}$, $\hat{v} = \begin{bmatrix} 2/\sqrt{22} \\ -3/\sqrt{22} \\ 3/\sqrt{22} \end{bmatrix}$

iii. $\hat{u} = \begin{bmatrix} 3/\sqrt{38} \\ 2/\sqrt{38} \\ -5/\sqrt{38} \\ 0 \end{bmatrix}$, $\hat{v} = \begin{bmatrix} -4/\sqrt{57} \\ 1/\sqrt{57} \\ -2/\sqrt{57} \\ 6/\sqrt{57} \end{bmatrix}$

d. $\|\vec{u}\|^2 + \|\vec{v}\|^2$

i. $5+52 = 57$

ii. $178+22 = 200$

iii. $38+57 = 95$

e. $\|\vec{u}+\vec{v}\|^2$

i. $\vec{u}+\vec{v} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$ $\quad \|\vec{u}+\vec{v}\| = \sqrt{9+64} = \sqrt{73}$ $\quad \|\vec{u}+\vec{v}\|^2 = 73$

ii. $\vec{u}+\vec{v} = \begin{bmatrix} 14 \\ 0 \\ -2 \end{bmatrix}$ $\quad \|\vec{u}+\vec{v}\| = \sqrt{196+4} = \sqrt{200}$ $\quad \|\vec{u}+\vec{v}\|^2 = 200$

iii. $\vec{u}+\vec{v} = \begin{bmatrix} -1 \\ 3 \\ -7 \\ 6 \end{bmatrix}$ $\quad \|\vec{u}+\vec{v}\| = \sqrt{1+9+49+36} = \sqrt{95}$ $\quad \|\vec{u}+\vec{v}\|^2 = 95$

f. $\left(\dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$

i. $\dfrac{8}{52}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \dfrac{2}{13}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8/13 \\ 12/13 \end{bmatrix}$

ii. $\dfrac{0}{22}\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = \vec{0}$

iii. $\dfrac{0}{57}\begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix} = \vec{0}$

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1g. $\|\vec{u}-\vec{v}\|$ =

i. $\vec{u}-\vec{v} = \begin{bmatrix} -5 \\ -4 \end{bmatrix}$ $\quad \|\vec{u}-\vec{v}\| = \sqrt{25+16} = \sqrt{41}$

ii. $\vec{u}-\vec{v} = \begin{bmatrix} 10 \\ 6 \\ -8 \end{bmatrix}$ $\quad \|\vec{u}-\vec{v}\| = \sqrt{100+36+64} = \sqrt{200}$

iii. $\vec{u}-\vec{v} = \begin{bmatrix} 7 \\ 1 \\ -3 \\ -6 \end{bmatrix}$ $\quad \|\vec{u}-\vec{v}\| = \sqrt{49+1+9+36} = \sqrt{95}$

2a. See 1g.

b. ii & iii are. (see 1a)

c. for ii & iii, $\pi/2$; for i. $\cos\theta = \frac{8}{\sqrt{5}\cdot 2\sqrt{13}} = \frac{4}{\sqrt{65}}$, $\theta = \cos^{-1}\left(\frac{4}{\sqrt{65}}\right) = 1.05165$ radians $\approx 60.255°$

d. See 1c.

e. See 1f.

f. for ii. & iii, use the unit vectors in 1f.

g. ii. $12a + 3b - 5c = 0$
$2a - 3b + 3c = 0$ $\quad$ rref $\Rightarrow \begin{bmatrix} 1 & 0 & -4/7 \\ 0 & 1 & -23/21 \end{bmatrix}$ $\quad w_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\quad W^{\perp} = \left\{ \begin{bmatrix} 3 \\ 23 \\ 21 \end{bmatrix} \right\}$

$x_1 = 4/7\, x_3$
$x_2 = 23/21\, x_3$ $\quad \Rightarrow \vec{w}_3 = \begin{bmatrix} 3 \\ 23 \\ 21 \end{bmatrix}$ $\quad \sqrt{9+529+441} = \sqrt{979}$
$x_3 = x_3$

iii. $3a + 2b - 5c + 0d = 0$
$-4a + b - 2c + 6d = 0$ $\quad$ rref $\Rightarrow \begin{bmatrix} 1 & 0 & -1/11 & -12/11 \\ 0 & 1 & -26/11 & 18/11 \end{bmatrix}$ $\quad \vec{w}_3 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

$x_1 = + 1/11\, x_3 + 12/11\, x_4$
$x_2 = 26/11\, x_3 - 18/11\, x_4$ $\quad \begin{bmatrix} 1 \\ 26 \\ 11 \\ 0 \end{bmatrix}$ $\quad w_4 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$
$x_3 = x_3$
$x_4 = x_4$

$3a + 2b - 5c + 0d = 0$
$-4a + b - 2c + 6d = 0$ $\quad$ rref $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -8/7 \\ 0 & 1 & 0 & 2/7 \\ 0 & 0 & 1 & -4/7 \end{bmatrix}$
$a + 26b + 11c + 0d = 0$

$x_1 = 8/7\, x_4$
$x_2 = -2/7\, x_4$ $\quad \begin{bmatrix} 8 \\ -2 \\ 4 \\ 7 \end{bmatrix}$
$x_3 = 4/7\, x_4$
$x_4 = x_4$

$$W^{\perp} = \left\{ \begin{bmatrix} 1 \\ 26 \\ 11 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -2 \\ 4 \\ 7 \end{bmatrix} \right\}$$

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3. $\vec{y}_{\parallel} = \frac{\vec{y}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u} = \frac{14+6}{49+1}\begin{bmatrix}7\\1\end{bmatrix} = \frac{2\cancel{0}}{5\cancel{0}}\begin{bmatrix}7\\1\end{bmatrix} = \begin{bmatrix}14/5\\2/5\end{bmatrix}$

$\vec{y}_{\perp} = \vec{y} - \vec{y}_{\parallel} = \begin{bmatrix}2\\6\end{bmatrix} - \begin{bmatrix}14/5\\2/5\end{bmatrix} = \begin{bmatrix}-4/5\\28/5\end{bmatrix}$

4. a. true
b. true
c. false. They are still orthogonal.
d. true
e. true if it is square
f. true
g. true

5. a i. $\vec{u}_1\cdot\vec{u}_2 = 0+0+0 = 0$ ✓ $\vec{u}_1\cdot\vec{u}_3 = 1+0-1=0$ ✓ $\vec{u}_2\cdot\vec{u}_3 = 0+0+0=0$ ✓
this is an orthogonal set.

ii. $\vec{u}_1\cdot\vec{u}_2 = 1-6+0+0+0 = -5$ X not orthogonal

iii. answers will vary: $\vec{v}_1 = \begin{bmatrix}3\\1\\0\\0\end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix}0\\-5\\-3\\0\end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix}5\\0\\-1\\0\end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix}0\\0\\4\\-5\end{bmatrix}$, etc.

6. i. $\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx = \int_{-\pi}^{\pi}\frac{1}{2}\left[\cos[(m+n)x] + \cos[(m-n)x]\right]dx$

cosine is even so $= \int_0^{\pi}\cos[(m+n)x] + \cos[(m-n)x]\,dx = \frac{\sin[(m+n)x]}{m+n} + \frac{\sin[(m-n)x]}{m-n}\Big|_0^{\pi} = 0$

the functions are orthogonal.

ii. $\int_{-1}^{1} 1(x)\,dx = 0$ since x is odd.

$\int_{-1}^{1} 1(1-3x^2)\,dx = 2\int_0^1 1-3x^2\,dx = 2[x - x^3]\Big|_0^1 = 2[1-1-0+0] = 2(0) = 0$
even

$\int_{-1}^{1} 1(-3x+5x^3)\,dx = 0$ since $-3x+5x^3$ is odd

$\int_{-1}^{1} x(1-3x^2)\,dx = \int_{-1}^{1} x-3x^3\,dx = 0$ since $x-3x^3$ is odd

56 ii. cont'd.

$$\int_{-1}^{1} x(-3x+5x^3)\,dx = \int_{-1}^{1} \underbrace{-3x^2+5x^4}_{\text{even}}\,dx = 2\left[-x^3+x^5\right]_0^1 = 2[-1+1-0+0]=0$$

$$\int_{-1}^{1} (1-3x^2)(-3x+5x^3)\,dx = \int_{-1}^{1} -3x+5x^3+9x^3-15x^5\,dx = 0$$ since function is odd.

iii. not $-3x+5x^3$ since that is given above.

$ax^4+bx^3+cx^2+dx+e = g(x)$. $(a\neq 0)$

$$\int_{-1}^{1}(ax^4+bx^3+cx^2+dx+e)\,1\,dx = \underbrace{\int_{-1}^{1} bx^3+dx\,dx}_{=0 \text{ odd}} + \int_{-1}^{1} ax^4+cx^2+e\,dx =$$
$$2\int_0^1 ax^4+cx^2+e\,dx = 2\left[\tfrac{a}{5}+\tfrac{c}{3}+e\right]_0^1$$

$$\int_{-1}^{1} x(ax^4+bx^3+cx^2+dx+e)\,dx = \int_{-1}^{1} ax^5+bx^4+cx^3+dx^2+ex\,dx =$$
$$\underbrace{\int_{-1}^{1} ax^5+cx^3+ex\,dx}_{=0 \text{ odd}} + \int_{-1}^{1} \underbrace{bx^4+dx^2}_{\text{even}}\,dx = 2\int_0^1 bx^4+dx^2\,dx = 2\left[\tfrac{b}{5}x^5+\tfrac{d}{3}x^3\right]_0^1$$

$$\boxed{=\frac{2b}{5}+\frac{2d}{3}=0} \qquad \boxed{\frac{2a}{5}+\frac{2c}{3}+2e=0}$$

$$\int_{-1}^{1}(1-3x^2)(ax^4+bx^3+cx^2+dx+e)\,dx = \int_{-1}^{1} ax^4+bx^3+cx^2+dx+e-3ax^6-3bx^5$$
$$-3cx^4-3dx^3-3ex^2\,dx = \underbrace{\int_{-1}^{1} bx^3+dx-3bx^5-3dx^3\,dx}_{=0 \text{ odd}} + \int_{-1}^{1} \underbrace{ax^4+cx^2+e-3ax^6-3cx^4}_{\text{even}}$$
$$-3ex^2\,dx = 2\int_0^1 ax^4+cx^2+e-3ax^6-3cx^4-3ex^2\,dx =$$

$$\boxed{\frac{2a}{5}+\frac{2c}{3}+2e-\frac{6a}{7}-\frac{6c}{5}-\frac{6e}{3}=0}$$

$6b+10d=0$

$6a+10c+30e=0$

$\left(\frac{2}{5}-\frac{6}{7}\right)a+\left(\frac{2}{3}-\frac{6}{5}\right)c+\cancel{\left(2-\frac{6}{3}\right)}e=0$ $\quad$ rref $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -35/3 \\ 0 & 1 & 0 & 5/3 & 0 \\ 0 & 0 & 1 & 0 & 10 \end{bmatrix}$

$a = +35/3\,e$

$b = -5/3\,d \leftarrow$ if $e=0$ this is poly from above. we let $d=0$

$c = -10e$

$d = d$

$e = e$

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 35 \\ 0 \\ -30 \\ 0 \\ 3 \end{bmatrix}$$

$g(x) = 35x^4-30x^2+3$