202 Proof Set #1 key

1. $1^3+2^3+3^3+\cdots n^3=\sum_{i=1} i^3 = \frac{n^2(n+1)^2}{4}$

a) base case $\sum_{i=1}^{1} 1^3 = \frac{1^2(1+1)^2}{4} = 1$ works

b) next case

Suppose formula works for n.

$$1^3+2^3+\cdots+n^3+(n+1)^3 = \sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4}+(n+1)^3$$

$$= \frac{(n+1)^2}{4}\left[n^2+4(n+1)\right] = \frac{(n+1)^2}{4}\left[n^2+4n+4\right] = \frac{(n+1)^2(n+2)^2}{4}$$

which is what we expect for the formula substituting $n+1$ for n.

Therefore, by mathematical induction, $1^3+2^3+3^2+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$. //

2. $(a+b)^3 = a^3+b^3$? if $a\neq 0, b\neq 0$

Suppose $a=2, b=3$

$(a+b)^3 = (2+3)^3 = 5^3 = 125$, but

$2^3+3^3 = 8+27 = 35$. $125 \neq 35$ Therefore, the expression

$(a+b)^3 = a^3+b^3$ is false. //

3. $\left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right]$ is our augmented matrix. To obtain a condition on a unique solution we need to put the matrix in echelon form.

$\left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right]$ $\frac{1}{a}R_1 \to R_1$ $\left[\begin{array}{cc|c} 1 & b/a & e/a \\ c & d & f \end{array}\right]$ $-cR_1+R_2 \to R_2$ $\left[\begin{array}{cc|c} 1 & b/a & e/a \\ 0 & d-cb/a & f-ce/a \end{array}\right]$

$aR_2 \to R_2$ $\left[\begin{array}{cc|c} 1 & b/a & e/a \\ 0 & ad-bc & fa-ce \end{array}\right]$ for this matrix to have a unique solution

a_{22} entry must be non-zero at this step, thus the condition is

$ad-bc\neq 0$, or $ad\neq bc$. //

6 cont'd.
invertible and no inverse exists. Then, by definition A is nonsingular. Which covers all cases. //

7. a. for all the problems in 7, let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ and let a_{ij}, b_{ij} and r, s be real numbers.

by definition of matrix addition, $A + B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$. Since each entry is real, by the commutative property of real numbers we can say $\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} = \begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} \\ b_{21}+a_{21} & b_{22}+a_{22} \end{bmatrix}$.

by definition of matrix addition this is equivalent to $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $= B + A$. //

b. $r(A+B) = r\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) = r\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$ by definition of matrix addition. Using the definition of scalar multiplication this becomes $\begin{bmatrix} r(a_{11}+b_{11}) & r(a_{12}+b_{12}) \\ r(a_{21}+b_{21}) & r(a_{22}+b_{22}) \end{bmatrix}$. using the distributive property of real numbers we get $\begin{bmatrix} ra_{11}+rb_{11} & ra_{12}+rb_{12} \\ ra_{21}+rb_{21} & ra_{22}+rb_{22} \end{bmatrix}$. by reversing the property of matrix addition this is equivalent to $\begin{bmatrix} ra_{11} & ra_{12} \\ ra_{21} & ra_{22} \end{bmatrix} + \begin{bmatrix} rb_{11} & rb_{12} \\ rb_{21} & rb_{22} \end{bmatrix}$ and then by definition of scalar multiplication this is $r\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + r\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ $= rA + rB$. //

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7c. $(r+s)A = (r+s)\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} (r+s)a_{11} & (r+s)a_{12} \\ (r+s)a_{21} & (r+s)a_{22} \end{bmatrix}$ by definition of scalar multiplication of matrices. Using the distributive property of real numbers on each component we get $\begin{bmatrix} ra_{11}+sa_{11} & ra_{12}+sa_{12} \\ ra_{21}+sa_{21} & ra_{22}+sa_{22} \end{bmatrix}$. Using matrix addition we obtain $\begin{bmatrix} ra_{11} & ra_{12} \\ ra_{21} & ra_{22} \end{bmatrix} + \begin{bmatrix} sa_{11} & sa_{12} \\ sa_{21} & sa_{22} \end{bmatrix}$ and by applying the definition of scalar multiplication to each matrix, we obtain $r\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + s\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = rA + sA.$ //

d. $(A^T)^T = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T\right)^T = \left(\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}\right)^T$ by applying the definition of the transpose, and then using it a second time we obtain $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A$. Thus $(A^T)^T = A$. //

e. $(A+B)^T = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right)^T = \left(\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}\right)^T$ by definition of matrix addition. Applying the transpose, we get $\begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{bmatrix}$ and by property of matrix addition this is equivalent to $\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} = A^T + B^T$. //

8. Let $A = a_{ij}$ and $B = b_{jk}$. Then the product of $AB = C$ is given by $C = c_{ik} = \sum a_{ij} b_{jk}$. Then $(AB)^T = C^T = c_{ki} = \sum b_{kj} a_{ji}$. Consider $B^T A^T$ where $B^T = b_{kj}$ and $A^T = a_{ji}$. Thus $B^T A^T = D = d_{ki} = \sum b_{kj} \cdot a_{ji}$. Thus since each entry is identical $C = D$ or $(AB)^T = B^T A^T$. //

9. Let $A = a_{ij}$ and $B = b_{jk}$ and $C = c_{jk}$. $A(B+C)$ means that each entry of $B+C$ is $b_{jk}+c_{jk}$ and the product $A(B+C)$, each entry is $\sum a_{ij}(b_{jk}+c_{jk})$ since these are real numbers, we can distribute to obtain $\sum(a_{ij}b_{jk} + a_{ij}c_{jk}) = \sum a_{ij}b_{jk} + \sum a_{ij}c_{jk}$ by commutativity and assocativity. These expressions are equivalent to $AB + AC$. //

10. Suppose $A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$. since A is diagonal $A^k = \begin{bmatrix} a_{11}^k & 0 \\ 0 & a_{22}^k \end{bmatrix}$

Thus $\sum_{k=0}^{\infty} \frac{A^k}{k!} = e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \dots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} a_{11}^2 & 0 \\ 0 & a_{22}^2 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} a_{11}^3 & 0 \\ 0 & a_{22}^3 \end{bmatrix} + \dots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} + \begin{bmatrix} \frac{a_{11}^2}{2} & 0 \\ 0 & a_{22}^2/2 \end{bmatrix} + \begin{bmatrix} \frac{a_{11}^3}{6} & 0 \\ 0 & \frac{a_{22}^3}{6} \end{bmatrix} + \dots$

by scalar multiplication, and then adding corresponding terms we get

$$\begin{bmatrix} 1 + a_{11} + \frac{1}{2}a_{11}^2 + \frac{1}{6}a_{11}^3 + \dots & 0 \\ 0 & 1 + a_{22} + \frac{1}{2}a_{22}^2 + \frac{1}{6}a_{22}^3 + \dots \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{a_{11}^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{a_{22}^k}{k!} \end{bmatrix} = \begin{bmatrix} e^{a_{11}} & 0 \\ 0 & e^{a_{22}} \end{bmatrix}. //$$