202 Proof Set #3 key

1. Consider the base case $T_1(\vec{v}) = A_1\vec{v}$. The composition $T_2(T_1(\vec{v})) = A_2(A_1\vec{v})$
$= A_2 A_1 \vec{v}$.

Suppose the composition formula applies to the first n linear transformations i.e. $T(\vec{v}) = T_n(T_{n-1}(T_{n-2}(\cdots(T_2(T_1(\vec{v})))))) = A_n A_{n-1} \cdots A_2 A_1$. we wish to show it it true for the next composition $T_{n+1}(T(\vec{v})) = T_{n+1}(T_n(\cdots T_2(T_1(\vec{v})))) = A_{n+1}(A_n(A_{n-1}\cdots A_2 A_1))$.

Thus it has been shown. //

2. Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

a. $\vec{u}\cdot\vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = [u_1\ u_2\ u_3]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3$ by definition of the dot product. Using the property of commutativity of multiplication of real numbers we can rearrange to get $v_1u_1 + v_2u_2 + v_3u_3 = [v_1\ v_2\ v_3]\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \vec{v}\cdot\vec{u}$. //

b. $(\vec{u}+\vec{v})\cdot\vec{w} = \left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right)\cdot\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}^T \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = [u_1+v_1\ \ u_2+v_2\ \ u_3+v_3]\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

$= (u_1+v_1)w_1 + (u_2+v_2)w_2 + (u_3+v_3)w_3 = u_1w_1 + v_1w_1 + u_2w_2 + v_2w_2 + u_3w_3 + v_3w_3$
by definition of the dot product   using distributive property of real numbers

Collecting terms we get $(u_1w_1 + u_2w_2 + u_3w_3) + (v_1w_1 + v_2w_2 + v_3w_3) =$
$[u_1\ u_2\ u_3]\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + [v_1\ v_2\ v_3]\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ by property of the dot product and this equals $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^T \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}^T \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \vec{u}\cdot\vec{w} + \vec{v}+w.$ //

c. for a real number, $(c\vec{u})\cdot\vec{v} = c\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^T \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = c[u_1\ u_2\ u_3]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} =$
$[cu_1\ cu_2\ cu_3]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = cu_1v_1 + cu_2v_2 + cu_3v_3$ by property of the dot product

2c cont'd.

factoring out the c we get $c(u_1v_1 + u_2v_2 + u_3v_3) = c(\vec{u}\cdot\vec{v})$.

rearranging the same expression $cu_1v_1 + cu_2v_2 + cu_3v_3 = u_1(cv_1) + u_2(cv_2) + u_3(cv_3)$ by commutativity of real numbers. ~~this~~ this expression equals

$$[u_1\ u_2\ u_3]\begin{bmatrix} cv_1 \\ cv_2 \\ cv_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^T \begin{bmatrix} cv_1 \\ cv_2 \\ cv_3 \end{bmatrix} = \vec{u}\cdot\left(c\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \vec{u}\cdot(c\vec{v}). \ //$$

d. $\vec{u}\cdot\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = [u_1\ u_2\ u_3]\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = u_1u_1 + u_2u_2 + u_3u_3 = u_1^2 + u_2^2 + u_3^2$.
by definition of dot product.

Since this is the sum of squares of real numbers, the terms are all positive and thus the sum is positive or zero, $\vec{u}\cdot\vec{u} \geq 0$. The only way this sum can equal zero is if every component u_i is equal to zero, which implies $\vec{u} = \vec{0}$.

3. Suppose that $\vec{y}$ is orthogonal to $\vec{u}$ and $\vec{v}$, Then $\vec{y}\cdot\vec{u} = 0$ and $\vec{y}\cdot\vec{v} = 0$. Adding these expressions we get $\vec{y}\cdot\vec{u} + \vec{y}\cdot\vec{v} = \vec{0}$, which implies $\vec{y}\cdot(\vec{u}+\vec{v}) = 0$ by the distributive property proved above.

4. let $\vec{x} \in W$ and $x \in W^\perp$. Since all the elements in W ~~and~~ orthogonal to elements in $W^\perp$, consider $\vec{y} \in W$ and $\vec{z} \in W^\perp$. then since $\vec{x}$ in W, $\vec{x}\cdot\vec{z} = 0$ and $\vec{x} \in W^\perp$, then $\vec{y}\cdot\vec{x} = \vec{0}$. But if we add $\vec{y}$ and $\vec{z}$ to represent and vector in the vector space containing W and $W^\perp$. then $\vec{x}\cdot z + \vec{x}\cdot\vec{y} = \vec{x}(\vec{y}+\vec{z}) = 0$ for any vector in the vector space. But if $\vec{x}$ is orthogonal to every vector in the vector space, it must be $\vec{0}$. //

5. Since every vector in $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_p\}$ is mutually orthogonal, they are also linearly independent and span the subspace. The vectors in $W^\perp$ are orthogonal to every vector in W and since $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_q\}$ forms an orthogonal basis for $W^\perp$, these vectors are linearly independent. Moreover, the ~~are~~ independent of all the vectors in W by the fact that they are orthogonal. By definition of $W^\perp$, every vector in $\mathbb{R}^n$, must be capable of being written

5 cont'd

as the sum of vectors in W and $W^{\perp}$, i.e. for some $\vec{x}$ in W and $\vec{y}$ in $\overline{W}$, every $\vec{z}$ in $\mathbb{R}^n$ can be written as $\vec{x}+\vec{y}$, so the vectors in $\{w_1, \dots w_p, v_1, \dots w_q\}$ must span $\mathbb{R}^n$, and since they are linearly independent must form a basis for $\mathbb{R}^n$. //

6. Consider the case where the dimension of V and W is the same: $\dim V = \dim W = n$. Then the matrix of the transformation $T: V \to W$ is $n \times n$ given by A. In this scenario, T could be an isomorphism since the $\dim \text{Col } A = n = \dim W$ since its possible for A to have a pivot in every row (and be onto). Consider the case where the $\dim V > \dim W$ let $\dim V = n$, and $\dim W = m$. Then the matrix representing $T: V \to W$ is given by the matrix A which is $m \times n$ w/ $m < n$. It is possible for there to be a pivot in every row but not in every column, so T can be onto but not one-to-one. The third possibility is that $\dim V < \dim W$. If $\dim V = n$ and $\dim W = m$, the matrix of the transformation $T: V \to W$ is given by the matrix A which is $m \times n$, but w/ $m > n$. Now we can have a pivot in every column, but since there are more rows than columns, it's not possible to have a pivot in every row and so T cannot be onto. Thus, $T: V \to W$ can be onto if $\dim V \geq \dim W$, but cannot be onto if $\dim W > \dim V$. //

7. If T is a linear transformation that maps $R^n \to R^n$, and is onto, consider the matrix of the transformation given by A, an $n \times n$ matrix. If A is onto, there is a pivot in every row and since there are the same number of columns, there is a pivot in every column. Thus, A is also one-to-one. Since A is both one-to-one and onto and is square, there exists a unique inverse given by $T^{-1}: R^n \to R^n$, and defined by the matrix A^{-1}. Since A^{-1} is invertible T^{-1} is also one-to-one and onto. //

8. Consider $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = k\vec{u} = \begin{bmatrix} ku_1 \\ ku_2 \\ \vdots \\ ku_n \end{bmatrix}$. Then $\|\vec{u}+\vec{v}\| = \left\| \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix} \right\|$

$\sqrt{(u_1+v_1)^2 + (u_2+v_2)^k + \dots + (u_n+v_n)^2}$ by definition of the norm. if $\vec{v} = k\vec{u}$ this can be written as $\sqrt{(u_1+ku_1)^2 + (u_2+ku_2)^2 + \dots + (u_n+ku_n)^2}$. pulling out the u_i from each term we get $\sqrt{u_1^2(1+k)^2 + u_2^2(1+k)^2 + \dots + u_n^2(1+k)^2} =$ $(1+k)\sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$. by definition of the norm, this is the same as $(1+k)\|\vec{u}\|$. using the distributive property of vectors $(1+k)\|\vec{u}\| = \|\vec{u}\| + k\|\vec{u}\|$ $= \|\vec{u}\| + \|k\vec{u}\| = \|\vec{u}\| + \|\vec{v}\|$. Since equalities are equivalent in both directions $\|\vec{u}+\vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$ iff $\vec{v} = k\vec{u}$. //

9. We want to prove that $\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$. Using the definition of the norm $\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = (u_1+v_1)^2 + (u_2+v_2)^2 + \dots + (u_n+v_n)^2$ $+ (u_1-v_1)^2 + (u_2-v_2)^2 + \dots + (u_n-v_n)^2$. distributing out all parentheses we get $(u_1^2 + 2u_1v_1 + v_1^2) + (u_2^2 + 2u_2v_2 + v_2^2) + \dots + (u_n^2 + 2u_nv_n + v_n^2) + (u_1^2 - 2u_1v_1 + v_1^2)$ $+ (u_2^2 - 2u_2v_2 + v_2^2) + \dots + (u_n^2 - 2u_nv_n + v_n^2)$. Cancelling the pairs of $\pm 2u_iv_i$ terms, we are left with $(u_1^2+v_1^2) + (u_2^2+v_2^2) + \dots + (u_n^2+v_n^2) + (u_1^2+v_1^2) +$ $(u_2^2+v_2^2) + \dots + (u_n^2+v_n^2) = (2u_1^2 + 2v_1)^2 + (2u_2^2 + 2v_2^2) + \dots + (2u_n^2 + 2v_n^2)$. Collecting all the u_i and v_i terms we get $(2u_1^2 + 2u_2^2 + \dots 2u_n^2) + (2v_1^2 + 2v_2^2 + \dots 2v_n^2)$ $= 2(u_1^2 + u_2^2 + \dots + u_n^2) + 2(v_1^2 + v_2^2 + \dots + v_n^2) = 2\|\vec{u}\| + 2\|\vec{v}\|$ by definition of the norm //

10. If A and B are similar then there exists P such that $A = PBP^{-1}$. Consider $A^T = (PBP^{-1})^T = (P^{-1})^T B^T P^T$ by property of the transpose, and since $(P^{-1})^T = (P^T)^{-1}$, thus A^T is similar to B^T since $A^T = (P^T)^{-1} B^T P^T$ using the similarity transformation $(P^T)^{-1}$. //

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11. we need the following identities:

$$\cos(a)\cos(b) = \tfrac{1}{2}\left(\cos(a+b) + \cos(a-b)\right)$$
$$\sin(a)\sin(b) = \tfrac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$$
$$\sin(a)\cos(b) = \tfrac{1}{2}\left(\sin(a+b) + \sin(a-b)\right)$$

a. $\int_{-\pi}^{\pi} \sin(nx)\cdot\sin(mx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x - \cos(m+n)x\,dx = \frac{1}{2}\left[\frac{1}{n-m}\sin(n-m)x - \frac{1}{n+m}\sin(n+m)x\right]_{-\pi}^{\pi} = 0$ Thus $\sin nx$ and $\sin mx$ are orthogonal under the given inner product for n, m integers.

b. $\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x + \cos(n-m)x\,dx = \frac{1}{2}\left[\frac{\sin(n+m)x}{n+m} + \frac{\sin(n-m)x}{n-m}\right]_{-\pi}^{\pi}$ $= 0$ thus $\cos nx$ and $\cos mx$ are orthogonal for n, m integers.

c. $\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)x + \sin(n-m)x\,dx = \frac{1}{2}\left[\frac{-\cos(n+m)x}{n+m} - \frac{\cos(n-m)}{n-m}\right]_{-\pi}^{\pi} = -\frac{1}{2}\cdot\frac{1}{n+m}\left(\cos(n+m)\pi - \cos(n+m)(-\pi)\right) - \frac{1}{2}\cdot\frac{1}{n-m}\left(\cos(n-m)\pi - \cos(n-m)(-\pi)\right)$

Since $\cos x$ is even $\cos k\pi$ and $\cos -k\pi$ are the same value, so we have $-\frac{1}{2}\cdot\frac{1}{n+m}(0) - \frac{1}{2}\cdot\frac{1}{n-m}(0) = 0$. Thus, these functions are orthogonal under the inner product. //

12. Consider $T: V \to W$ w/ $\vec{x}, \vec{y}$ in V. Then, by definition $T(\vec{x}), T(\vec{y})$ in W, and specifically in range of T. Since T is linear, we know that $T(\vec{x}) + T(\vec{y}) = T(\vec{x}+\vec{y})$, but this also means that since $T(\vec{x})$ and $T(\vec{y})$ in range of T, and $T(\vec{x}+\vec{y})$ in range of T since $\vec{x}+\vec{y}$ in V, this means that $T(\vec{x}) + T(\vec{y})$ is in the space range T. Since $k\vec{x}$ in V, and since we know that $kT(\vec{x}) = T(k\vec{x})$ by property of the linear transformation, this means that $kT(\vec{x})$ is in range T. Finally, since $T(\vec{0}) = \vec{0}$ in W, we know $\vec{0}$ is in the range of T. Thus the range of T is a subspace of W. //

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13. If $P^{-1} = P^T$ then $P^{-1}P = P^TP$ implies $I = P^TP$. which is a property of orthogonal matrices. Thus the columns of P form an orthogonal basis for $\mathbb{R}^n$. If the columns of P form an orthogonal basis for $\mathbb{R}^n$, then the matrix is invertible. Moreover, $P^TP = I$, thus since $P^{-1}P = I$, this implies $P^{-1} = P^T$. //

14. The differential operator $\frac{d}{dx}$ is a linear transformation on $C(-\infty, \infty)$. To prove this, we check 1) does a $\vec{0}$ exist, 2) is the sum in the set, 3) is the scalar multiple in the set.

1) $\frac{d}{dx}[\vec{0}] = \vec{0}$ Thus ~~if~~ for $\frac{d}{dx}[f] = T(f)$, $T(\vec{0}) = \vec{0}$

2) $\frac{d}{dx}[f+g] = \frac{d}{dx}[f] + \frac{d}{dx}[g]$ corresponds to $T(f+g) = T(f) + T(g)$.

3) $\frac{d}{dx}[kf] = k\frac{d}{dx}[f]$ corresponds to $T(kf) = kT(f)$. Thus $\frac{d}{dx}$ is a linear transformation on $C(-\infty, \infty)$.