MAT 212, Exam #1, Summer 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Verify that $y = \frac{1}{1+x^2}$, is a solution to $y' + 2xy^2 = 0$. (6 points)

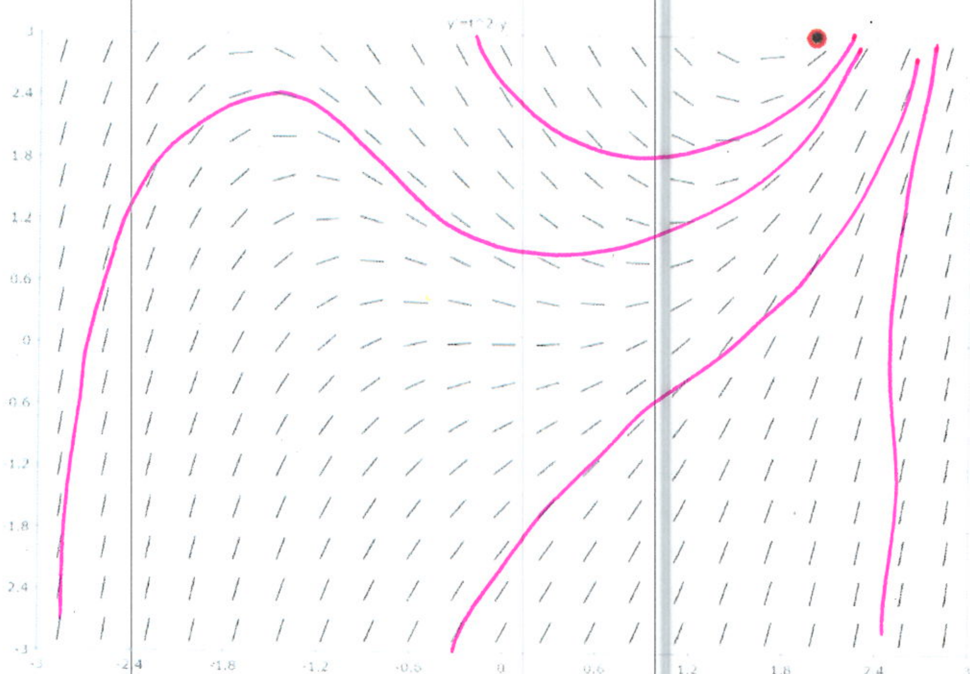
$y = (1+x^2)^{-1}$

$y' = -1(1+x^2)^{-2} \cdot 2x$

$= \frac{-2x}{(1+x^2)^2}$

$\frac{-2x}{(1+x^2)^2} + 2x\left(\frac{1}{1+x^2}\right)^2 = 0$ ✓

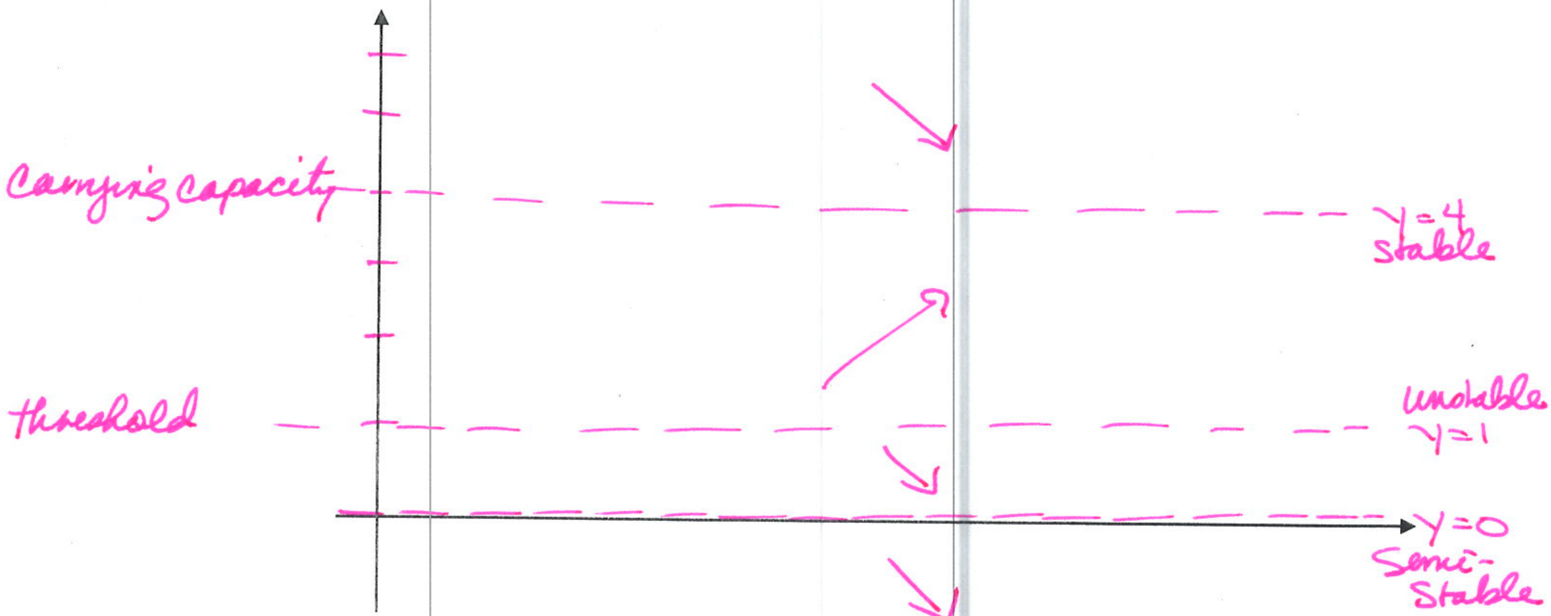
2. Shown below is the slope field for $\frac{dy}{dx} = x^2 - y$. Plot at least 4 sample trajectories with different behaviours. Track the path forward and backward in time (so that the path begins and ends on the edges of the graph. (6 points)



3. Sketch the slope field for the autonomous equation $y' = -y^2(y-1)(y-4)$. Label each equilibrium and classify it as stable, semi-stable or unstable. Are any of them a carrying capacity or threshold? Draw the phase plane. (12 points)

Direction Field

$y=0,\ y=1,\ y=4$



Phase Plane

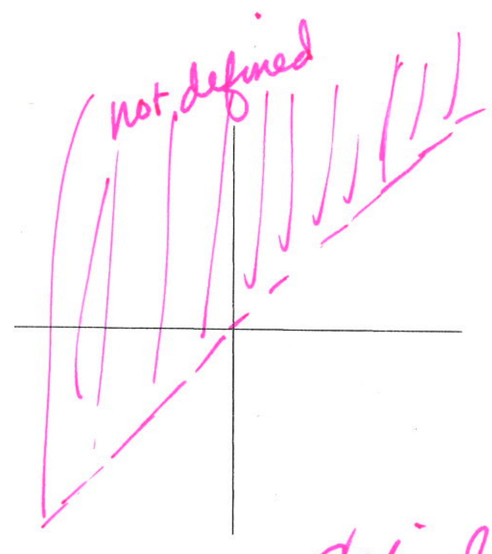
4. Sketch the region in the plane where a solution to the ODE $\frac{dy}{dt} = \sqrt{t-y}$ is guaranteed to exist. Be sure to check all conditions and show your work. (8 points)

$t-y \geq 0 \quad t \geq y$ or $y \leq t$

$f = (t-y)^{1/2}$

$\frac{\partial f}{\partial y} = \frac{1}{2}(t-y)^{-1/2}(-1)$

$= \frac{-1}{2\sqrt{t-y}}$ undefined at $t=y$ also



defined for $y<t$

5. Solve $\frac{dy}{dx} = \frac{(x-1)y^5}{x^2(2y^3-y)}$. (10 points)

$$\frac{(2y^3-y)}{y^5}\,dy = \frac{(x-1)}{x^2}\,dx$$

$$-\frac{2}{y} + \frac{1}{3y^3} = \ln x + \frac{1}{x} + C$$

$$\int \frac{2}{y^2} - \frac{1}{y^4}\,dy = \int \frac{1}{x} - \frac{1}{x^2}\,dx$$

$$2y^{-2} - y^{-4}$$

6. Classify each differential equation as i) linear or nonlinear, ii) ordinary or partial, iii) its order. (3 points each)

a. $yy' = x(y^2 + 1)$

nonlinear, first order, ordinary

b. $\frac{d^4y}{dx^4} = y \cos x$

ordinary, linear, 4th order

c. $2\sqrt{x}\frac{\partial u}{\partial x} + \frac{\partial u}{dy}\cos^2 y = 2xyu$

partial, linear, 1st order

d. $u_{xy} + u_{xx} = e^u \tan x \sec y$

partial, 2nd order, nonlinear

7. A 1000L tank initially contains only pure water. A hose begins adding to the tank at a rate of 5L/min with a concentration of iodine salt of 40g/L. The well-mixed solution flows out of the tank at a rate of 6L/min. Find an equation that models the amount of iodine in the tank after time t. Find the maximum amount of iodine in the tank (if one exists). (10 points)

$$\frac{dA}{dt} = \frac{5L}{min} \cdot \frac{40g}{L} - \frac{A}{(1000-t)L} \cdot \frac{6L}{min}$$

$$A(0) = 0$$

$$A' + \frac{6A}{1000-t} = 200$$

$$\mu = e^{\int \frac{6}{1000-t} dt} = e^{\int \frac{-6}{t-1000} dt} = e^{-6 \ln(t-1000)} = \frac{1}{(t-1000)^{6}}$$

$$(t-1000)^{-6} A' + 6(t-1000)^{-7} A = 200(t-1000)^{-6}$$

$$\int \left((t-1000)^{-6} A\right)' = \int 200(t-1000)^{-6} dt$$

$$(t-1000)^{-6} A = \frac{200(t-1000)^{-5}}{-5} + C = \frac{-40}{(t-1000)^5} + C$$

$$A = -40(t-1000) + C(t-1000)^6$$

$$0 = -40(0-1000) + C(0-1000)^6 \Rightarrow C = -4 \times 10^{-14}$$

$$A(t) = 40(1000-t) - 4\times10^{-14}(t-1000)^6$$

max at

$t = 301.$ min

$A = 25,294g$

8. Use the method of integrating factors to find the particular solution for $xy' = 2y + x^3 \cos x$, $y(\pi) = 0$. (10 points)

$$y' - \frac{2}{x} y = x^2 \cos x \qquad \mu = e^{\int -\frac{2}{x} dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

$$x^{-2} y' - 2x^{-3} y = \cos x$$

$$\int (x^{-2} y)' = \int \cos x \, dx$$

$$\frac{y}{x^2} = \sin x + C$$

$$y = x^2 \sin x + Cx^2$$

$$0 = \pi^2 (0) + C(\pi^2) \Rightarrow C = 0$$

$$\boxed{y = x^2 \sin x}$$

9. Rewrite the equation $y' + \frac{6}{x} y = 3y^{\frac{4}{3}}$ as a linear equation. [You do not need to solve.] (8 points)

$$n = 4/3$$

$$(1 - 4/3) y^{-4/3} = -\frac{1}{3} y^{-4/3}$$

$$z = y^{-1/3}$$

$$z' = -\frac{1}{3} y^{-4/3} y'$$

$$-\frac{1}{3} y^{-4/3} y' - \frac{2}{x} y^{-1/3} = -1$$

$$\boxed{z' - \frac{2}{x} z = -1}$$

10. Verify that the equation $(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0$ is exact. Then find the general solution. (10 points)

M N

$\frac{\partial M}{\partial y} = e^{xy} + xye^{xy}$ ✓

$\frac{\partial N}{\partial x} = e^{xy} + yxe^{xy}$

$\varphi: x + y^2 + e^{xy} = K$

$\int 1 + ye^{xy}\,dx = x + e^{xy} + f(y)$

$\int 2y + xe^{xy}\,dy = y^2 + e^{xy} + g(x)$

11. Find the general solution to $(x^2 - y^2)y' = 2xy$. (10 points)

$y' = \frac{2xy}{x^2 - y^2}$ homogeneous of order 2

$y = vx$

$y' = v'x + v$

$v'x + v = \frac{2x \cdot vx}{x^2 - v^2x^2} = \frac{\cancel{x^2}(2v)}{\cancel{x^2}(1 - v^2)}$

$v'x = \frac{2v}{1-v^2} - \frac{v(1-v^2)}{1-v^2} = \frac{2v - v + v^3}{1-v^2} = \frac{v(1+v^2)}{1-v^2}$

$v + v^3$

$\frac{1-v^2}{v(1+v^2)} \cdot dv = \frac{1}{x}dx$

$\frac{A}{v} + \frac{Bv + C}{1+v^2} = \frac{A + Av^2 + Bv^2 + Cv}{v(1+v^2)} = \frac{1-v^2}{v(1+v^2)}$

$A + B = -1$ $B = -2$

$C = 0$

$A = 1$

$\frac{y}{x} = v$

$\int \frac{1}{v} - \frac{2v}{1+v^2}\,dv = \int \frac{1}{x}dx$

$\ln v - \ln|1 + v^2| = \ln x + C$

$\cancel{\ln}\left|\frac{v}{1+v^2}\right| = \cancel{\ln}(Ax)$

$\frac{v}{1+v^2} = Ax$

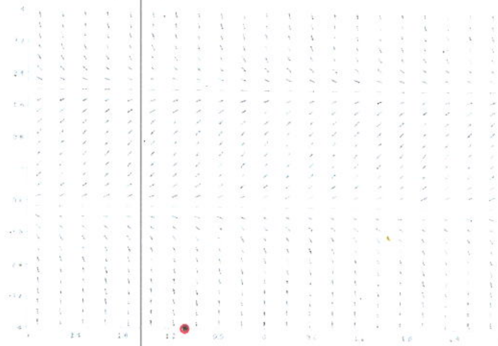
$\boxed{\frac{(y/x)}{1 + (y/x)^2} = Ax}$

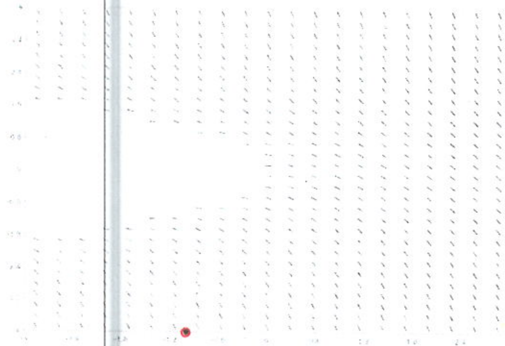
12. We want to approximate the solution to $y' = x + \sqrt[3]{y}$ at the point $x = 3$ in 10 steps. Given that $y(0) = 1$, compute the first 3 steps of the approximation with Euler's method. (12 points)

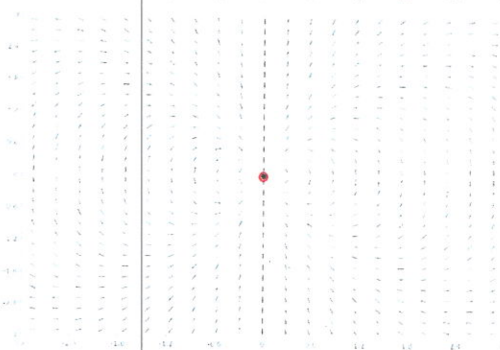
$\frac{3-0}{10} = .3$

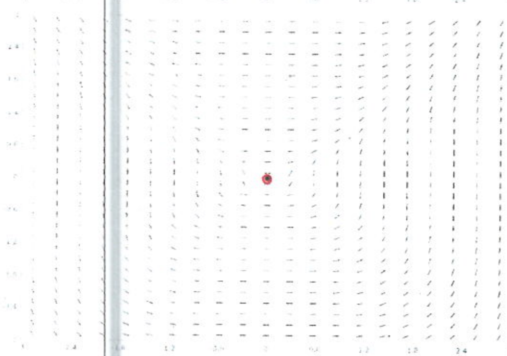
n	x_n	y_n	$f(x_n, y_n)$	$y_{n+1} = y_n + .3(f_n)$
0	0	1	$0 + \sqrt[3]{1} = 1$	$1 + .3(1) = 1.3$
1	.3	1.3	$.3 + \sqrt[3]{1.3} = 1.39139$	$1.3 + .3(1.39139) = 1.717417865$
2	.6	1.717	$.6 + \sqrt[3]{1.717} = 1.7975$	$1.717 + .3(1.7975) = 2.256681489 = y_3$
3	.9	2.25		

13. For each of the direction fields shown below, match the field to the differential equation that produced it. (12 points)

a. 

c. 

b. 

d. 

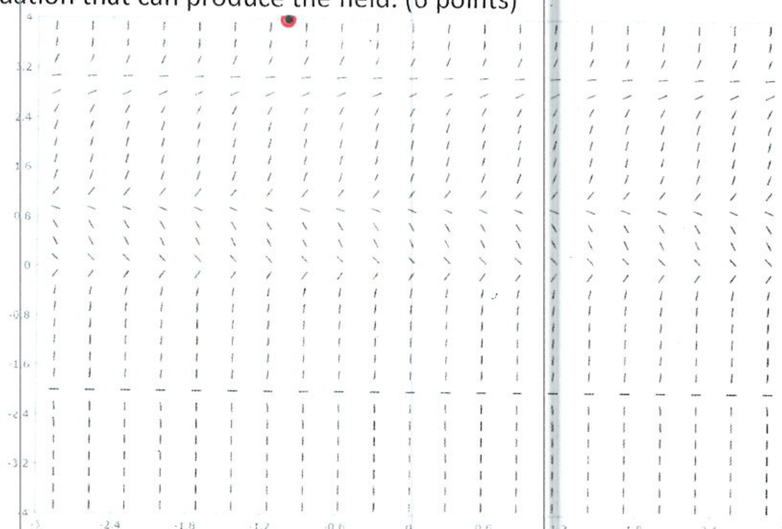
i. $y' = \frac{t^3}{y^2}$ D

ii. $y' = -(y+1)(y-2)$ A

iii. $y' = -\sqrt{t+y^2}$ C

iv. $y' = \frac{y^2-t^2}{ty}$ B

14. Consider the slope field shown below. If the equilibria are assumed to be integer values, write a differential equation that can produce the field. (6 points)



y=3 semistable

y=1 unstable

y=0 stable

y=-2 semi-stable? or unstable?

$y' = y(y-1)(y-3)^2(y+2)$

or $y(y-1)(y-3)^2(y+2)^2$