212 Homework #1 Key

1a. $y = te^t$; $y' = e^t + te^t$ (product rule)

b. $y = e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}$; $y' = 2te^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}\cdot e^{-t^2} + 2te^{t^2}$

$= 2te^{t^2}\int_0^t e^{-s^2}\,ds + 1 + 2te^{t^2}$ (product rule + chain rule + 2nd fundamental theorem of calculus)

c. $y = e^t\cos 2t$; $y' = e^t\cos 2t - 2e^t\sin 2t$ (product rule + chain rule)

d. $y = (\cos t)\ln(\cos t) + t\sin t$; $y' = -(\sin t)\ln(\cos t) + (\cancel{\cos t})\frac{-\sin t}{\cancel{\cos t}}$

$+ \sin t + t\cos t = -(\sin t)\ln(\cos t) - \cancel{\sin t} + \cancel{\sin t} + t\cos t$

$= -(\sin t)\ln(\cos t) + t\cos t$ (product rule + chain rule)

2.a. $y_2(t) = \cosh t$
$y_2' = \sinh t$
$y_2'' = \cosh t$

$y_1(t) = e^t$
$y_1' = e^t$
$y_1'' = e^t$

$y'' - y = 0$
$\cosh t - \cosh t = 0$ ✓
$e^t - e^t = 0$ ✓

b. $y_1(t) = t/3$
$y_1' = 1/3$
$y_1'' = 0$
$y_1''' = 0$
$y_1^{iv} = 0$

$y_2(t)\ \frac{t}{3} + e^{-t}$
$y_2' = 1/3 - e^{-t}$
$y_2'' = e^{-t}$
$y_2''' = -e^{-t}$
$y_2^{iv} = e^{-t}$

$y^{iv} + 4y''' + 3y = t$
$0 + 0(4) + 3(t/3) = t$ ✓
$e^{-t} + 4(-e^{t}) + 3(\frac{t}{3} + e^{-t}) =$
$e^{-t} - 4e^{-t} + 3e^{-t} + t = t$ ✓

c. $y_1(t) = 3t + t^2$
$y_1' = 3 + 2t^2$

$ty' - y = t^2$
$t(3 + 2t^2) - (3t + t^2) =$
$3t + 2t^2 - 3t - t^2 = t^2$ ✓

Homework #1 (cont'd)

3a. $y'' + y' - 6y = 0$; $y = e^{rt}$; $y' = re^{rt}$, $y'' = r^2e^{rt}$

$r^2e^{rt} + re^{rt} - 6e^{rt} = e^{rt}(r^2 + r - 6) = 0$ $\quad e^{rt} \neq 0 \quad r^2 + r - 6 = 0$

$(r+3)(r-2) = 0 \quad r = -3, 2 \quad \Rightarrow y = e^{-3t}, e^{2t}$

b. $t^2y'' + 4ty' + 2y = 0$, $y = t^r$, $y' = rt^{r-1}$, $y'' = r(r-1)t^{r-2}$

$t^2[r(r-1)t^{r-2}] + 4t[rt^{r-1}] + 2t^r = 0$

$r(r-1)t^r + 4rt^r + 2t^r = [t^r]\cdot(r^2 - r + 4r + 2) = 0$ $\quad t^r \neq 0$ unless $t = 0$

$r^2 + 3r + 2$

$r^2 + 3r + 2 = 0 \quad (r+2)(r+1) = 0 \quad r = -2, -1$; $y = t^{-2}, t^{-1}$

4a. $\int \frac{t}{1-t}\,dt = \int \frac{-t}{t-1}\,dt$

$$\begin{array}{r} -1 \\ t-1 \enclose{longdiv}{-t} \\ \underline{+t \mp 1} \\ -1 \end{array}$$

$\int -1 - \frac{1}{t-1}\,dt$

(long division)

$= -t - \ln|t-1| + C$

b. $\int y \sin y^2\,dy \quad u = y^2$, $du = 2y\,dy$, $\frac{1}{2}du = y\,dy$

$\int \frac{1}{2}\sin u\,du = -\frac{1}{2}\cos u$

$\Rightarrow -\frac{1}{2}\cos y^2 + C$

(Substitution)

c. $\int \frac{1}{4+t^2}\,dt = \frac{1}{2}\arctan\left(\frac{t}{2}\right) + C$

d. $\int x^2e^x\,dx$

$u = x^2 \quad dv = e^x$
$du = 2x \quad v = e^x$

$x^2e^x - \int 2xe^x\,dx$

$u = x \quad dv = e^x$
$du = dx \quad v = e^x$

$x^2e^x - 2\left[xe^x - \int e^x\,dx\right] = x^2e^x - 2xe^x + 2e^x + C$

(by parts)

or (tabular method)

+/-	u	dv
+	x^2	e^x
-	$2x$	e^x
+	2	e^x
	0	e^x

$x^2e^x - 2xe^x + 2e^x + C$

4e. $\int \cos^2 t \, dt = \frac{1}{2}\int 1 + \cos 2t \, dt = \frac{1}{2}[t + \frac{1}{2}\sin 2t] + C$

$\frac{1}{2}t + \frac{1}{4}\sin 2t + C$ (trig identity)

5a. $1+i$ $\quad \|1+i\| = \sqrt{1^2+1^2} = \sqrt{2}$

$\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\, e^{\frac{\pi}{4}i}$

$\uparrow$ $\cos(\pi/4)$ $\quad \uparrow$ $\sin(\pi/4)$

$r = \sqrt{a^2+b^2}$

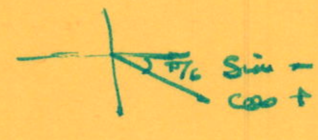
$\theta = \tan^{-1}\left(\frac{b}{a}\right) + k\pi \quad k = 0, 1$

$re^{\theta i} = r\cos\theta + ir\sin\theta$

b. $\sqrt{3} - i$ $\quad \|\sqrt{3} + i\| = \sqrt{3+1} = \sqrt{4} = 2$

$2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 2e^{-\frac{\pi}{6}i}$ or $2e^{\frac{11\pi}{6}i}$

$\cos\pi/6$ $\quad \sin\pi/6$



6. $\sinh x = \frac{e^x - e^{-x}}{2}$ $\quad \cosh x = \frac{e^x + e^{-x}}{2}$ $\quad (\sinh x)' = \frac{e^x + e^{-x}}{2} = \cosh x$

$\tanh x = \frac{\sinh x}{\cosh x}$ $\quad (\cosh x)' = \frac{e^x - e^{-x}}{2} = \sinh x$

$[\tanh x]' = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$

$= \operatorname{sech}^2 x$

$\int \sinh x \, dx = \int \frac{e^x - e^{-x}}{2} dx = \frac{1}{2}\int e^x - e^{-x} dx = \frac{1}{2}[e^x + e^{-x}] = \cosh x$

7. a. $\vec{u} - i\vec{v} = \begin{bmatrix} 5 \\ -i \end{bmatrix} - i\begin{bmatrix} 3i \\ 1-4i \end{bmatrix} = \begin{bmatrix} 5 \\ -i \end{bmatrix} + \begin{bmatrix} -3i^2 \\ -i+4i^2 \end{bmatrix} = \begin{bmatrix} 5 \\ -i \end{bmatrix} + \begin{bmatrix} -3(-1) \\ -i+4(-1) \end{bmatrix}$

$= \begin{bmatrix} 5 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ -4-i \end{bmatrix} = \begin{bmatrix} 8 \\ -5-i \end{bmatrix}$

b. $2\vec{x} + 3\vec{y} = 2\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 0 \\ i \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ 3i \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 6+3i \end{bmatrix}$

c. $A + 2B = \begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix} + 2\begin{bmatrix} i & 1+i \\ 2-i & -3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 2i & 2+2i \\ 4-2i & -6 \end{bmatrix} =$

$\begin{bmatrix} 2+2i & 1+2i \\ 8-2i & -5 \end{bmatrix}$

Homework#1 Cont'd (2&2)

7d. $\bar{B} = \begin{bmatrix} -i & 1-i \\ 2+i & -3 \end{bmatrix}$

8a. $\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n(x^n)}\right| = \left|\frac{n+1}{n} \cdot \frac{\cancel{2^n}}{\cancel{2^n}\cdot 2} \cdot \frac{\cancel{x^n}\cdot x}{\cancel{x^n}}\right| \Rightarrow$

$\lim_{n\to\infty} \left|\left(\frac{n+1}{n}\right)^{\to 1} \cdot \frac{x}{2}\right| = \left|\frac{x}{2}\right| < 1 \quad |x| < 2$ radius of convergence $= 2$

b. $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n^2}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2x+1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(2x+1)^n}\right| = \left|\frac{\cancel{(2x+1)^n}\cdot(2x+1)}{\cancel{(2x+1)^n}} \cdot \frac{n^2}{(n+1)^2}\right| \Rightarrow$

$\lim_{n\to\infty} \left|\left(\frac{n^2}{(n+1)^2}\right)^{\to 1} \cdot 2x+1\right| = \frac{|2x+1|}{2} < \frac{1}{2}$ $\quad |x+1/2| < 1/2$

radius of convergence $= 1/2$

$-1 < 2x+1 < 1$

$-1 \quad\quad -1 \quad -1$

$\frac{-2}{2} < \frac{2x}{2} < \frac{0}{2}$

$-1 < x < 0$

$\frac{0-(-1)}{2} = R = 1/2$

9.a. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ $\quad$ b. $\sum_{n=0}^{\infty} x^n$ $\quad$ c. $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ $\quad$ d. $a = x^3$

$\left[\sum_{n=0}^{\infty} x^n\right]' = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$ $\quad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ $\quad \left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$

$\left[(1-x)^{-1}\right]' = (1-x)^{-2}(-1)(-1)$

d. $x^3 \sum_{n=0}^{\infty} (n+1)x^n = \sum_{n=0}^{\infty} (n+1) x^{n+3}$