

Instructions: Show all work. Some problems will instruct you to complete operations by hand, some can be done in the calculator. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question.

1. Determine whether the transformations are linear. If they are, prove it. If they are not, find a counterexample.

a. $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{bmatrix}$ $KT\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = k\begin{bmatrix} \frac{a+b}{2} \\ \frac{a+b}{2} \end{bmatrix} = \begin{bmatrix} \frac{ka+kb}{2} \\ \frac{ka+kb}{2} \end{bmatrix}$

$T(\vec{0}) = \begin{bmatrix} \frac{0+0}{2} \\ \frac{0+0}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓ $T\left(\begin{bmatrix} ka \\ kb \end{bmatrix}\right) = \begin{bmatrix} \frac{ka+kb}{2} \\ \frac{ka+kb}{2} \end{bmatrix}$ ✓

$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) = \begin{bmatrix} \frac{a+b}{2} \\ \frac{a+b}{2} \end{bmatrix} + \begin{bmatrix} \frac{c+d}{2} \\ \frac{c+d}{2} \end{bmatrix}$, $T\left(\begin{bmatrix} a+c \\ b+d \end{bmatrix}\right) = \begin{bmatrix} \frac{a+c+b+d}{2} \\ \frac{a+c+b+d}{2} \end{bmatrix}$ ✓
 $= \begin{bmatrix} \frac{a+b+c+d}{2} \\ \frac{a+b+c+d}{2} \end{bmatrix}$ ✓ it is linear

b. $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y-z \\ 2xy \\ x+z+1 \end{bmatrix}$

not linear

$T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0+0-0 \\ 2(0)(0) \\ 0+0+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

fails other tests as well

2. Find the kernel and the range for $T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x+y-z+w \\ 2x+y+4z+w \\ 3x+y+9z \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 4 & 1 \\ 3 & 1 & 9 & 0 \end{bmatrix} \Rightarrow \text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$

range = \mathbb{R}^3 $\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}\right\}$

kernel = $\text{Span}\left\{\begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \end{bmatrix}\right\}$

$x_1 = -3/2 x_4$
 $x_2 = 0$
 $x_3 = 1/2 x_4$
 $x_4 = x_4$