MATH 240, Exam #1 Practice, Summer 2017 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Let $\vec{u} = \langle 1, 2, 4\rangle$ and $\vec{v} = \langle -1, 0, 8\rangle$. Find (10 points)

$$\frac{\vec{u} - 4\vec{v}}{\|\vec{u} - \vec{v}\|}$$

$\vec{u} - 4\vec{v} = \langle 1,2,4\rangle - 4\langle -1,0,8\rangle$

$= \langle 1,2,4\rangle + \langle 4,0,-32\rangle$

$= \langle 5, 2, -28\rangle$

$\vec{u} - \vec{v} = \langle 1,2,4\rangle - \langle -1,0,8\rangle$

$= \langle 2, 2, -4\rangle$

$\|\vec{u} - \vec{v}\| = \sqrt{2^2 + 2^2 + (-4)^2} =$

$\sqrt{4+4+16} = \sqrt{24}$

$= 2\sqrt{6}$

$= \dfrac{\langle 5, 2, -28\rangle}{2\sqrt{6}}$

2. Find the equation of the line that passes through the points $(2,2,6)$ and $(1,-3,4)$. Write your answer in: (8 points)
   a. Parametric form

$\vec{r}(t) = (2-t)\hat{i} + (2-5t)\hat{j} + (6-2t)\hat{k}$

$\vec{v} = \langle 1-2, -3-2, 4-6\rangle =$

$\langle -1, -5, -2\rangle$

   b. Symmetric form

$$\frac{x-2}{-1} = \frac{y-2}{-5} = \frac{z-6}{-2}$$

$x = 2-t$

$y = 2-5t$

$z = 6-2t$

3. Find a set of parametric equations of the line through the point $(1,0,-11)$ that is parallel to the line $x = 5 + t, \quad y = -5 + 6t, \quad z = 3.$ (8 points)

$\vec{v} = \langle 1, 6, 0 \rangle$

$x = 1 + t, \; y = 6t, \; z = -11$

4. Find the standard equation of the sphere that has $(3,1,-3)$ and $(1,5,7)$ as end points of the diameter. (8 points)

midpoint = center

$\left(\frac{3+1}{2}, \frac{1+5}{2}, \frac{-3+7}{2}\right) = \left(\frac{4}{2}, \frac{6}{2}, \frac{4}{2}\right) = (2,3,2)$

distance $= \sqrt{(3-1)^2 + (1-5)^2 + (-3-7)^2} =$ diameter

$\sqrt{2^2 + (-4)^2 + (-10)^2} = \sqrt{4 + 16 + 100} = \frac{\sqrt{120}}{2}$

$= \frac{2\sqrt{30}}{2} = \sqrt{30}$

$(x-2)^2 + (y-3)^2 + (z-2)^2 = 30$

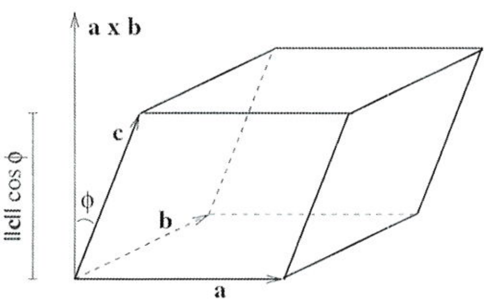
5. The projection of the vector $\vec{u}$ in the direction of $\vec{v}$ is given by $proj_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$. Find the projection of the vector $\vec{u} = \langle 3, -2, 9\rangle$ in the direction of the vector $\vec{v} = \langle 7 - 3, 4\rangle$. (8 points)

$\vec{u}\cdot\vec{v} = 21 + 6 + 36 = 63$

$\vec{v}\cdot\vec{v} = 49 + 9 + 16 = 74$

$proj_{\vec{v}}\vec{u} = \frac{63}{74}\langle 7, -3, 4\rangle$

6. Consider the parallelepiped (slanted box) determined by the position vectors $\vec{a} = \langle 3, -5, 6\rangle$, $\vec{b} = \langle 7, 1, 1\rangle$, $\vec{c} = \langle 4, 3, 1\rangle$. The volume is given by the triple scalar product $\left|\vec{c}\cdot(\vec{a}\times\vec{b})\right| = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$. Find the volume of the parallelepiped. (10 points)



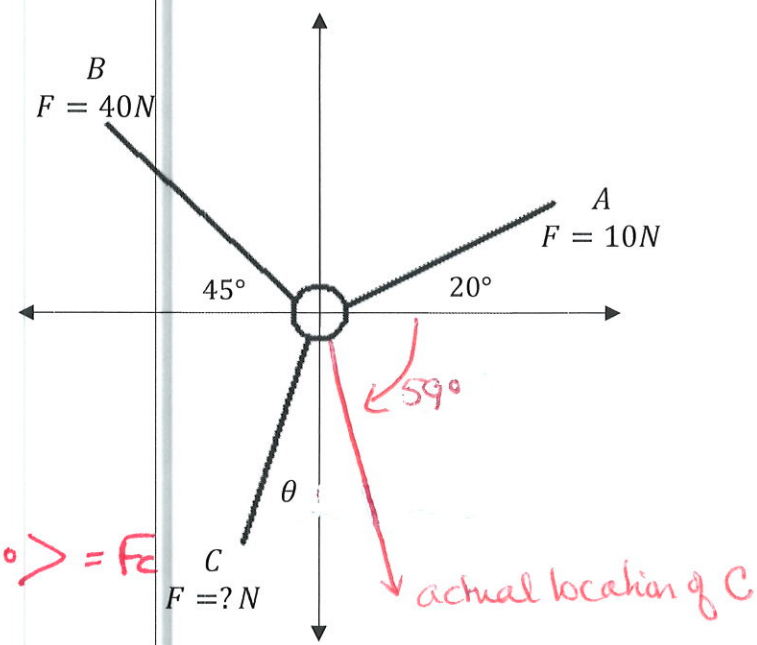
$\begin{vmatrix} 3 & -5 & 6 \\ 7 & 1 & 1 \\ 4 & 3 & 1 \end{vmatrix} =$

$3\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 5\begin{vmatrix} 7 & 1 \\ 4 & 1 \end{vmatrix} + 6\begin{vmatrix} 7 & 1 \\ 4 & 3 \end{vmatrix} =$

$3(1-3) + 5(7-4) + 6(21-4) =$

$3(-2) + 5(3) + 6(17) = -6 + 15 + 102 = 111$

7. Three people located at A, B, C pull on ropes tied to a ring. If A and B are pulling as shown, find the magnitude and direction with which C must pull so that no one moves (system is in equilibrium). You may round your final answer to 2 decimal places. (12 points)



$F_A = \langle 10\cos 20°, 10\sin 20° \rangle$

$F_B = \langle 40\cos 135°, 40\sin 135° \rangle$

$F_C = ?$ $\quad F_A + F_B + F_C = 0$

$F_A + F_B = \langle -18.887, 31.70447 \rangle$

$\|F_A + F_B\| \approx 36.9038$

$\langle 36.9038\cos\theta°, 36.9038\sin\theta° \rangle = F_C$

$\vec{F}_C = \langle 18.887, -31.704 \rangle \leftarrow$ this vector in QIV

$\langle 18.89, -31.70 \rangle$ $\quad \theta = \tan^{-1}\left(\frac{-31.7044}{18.887}\right) = -59°$

$F_C = 36.9N$ at 300.8° (roughly 301°)

8. Consider a particle traveling along a path determined by $\vec{r}(t) = \sin t\,\hat{\imath} + 2\cos t\,\hat{\jmath} + 2t\hat{k}$.

a. Find the velocity of the object. (3 points)

$\vec{r}'(t) = \cos t\,\hat{\imath} - 2\sin t\,\hat{\jmath} + 2\hat{k}$

b. What is the speed of the object? (3 points)

$\|\vec{r}'(t)\| = \sqrt{\cos^2 t + 4\sin^2 t + 4}$

$= \sqrt{5 + 3\sin^2 t}$

c. What is the acceleration of the object? (3 points)

$$\vec{r}''(t) = -\sin t\,\hat{i} - 2\cos t\,\hat{j} + 0\hat{k}$$

9. A volleyball is hit when it is 3 feet off the ground and 10 feet from a 6-foot-high net. It leaves the point of impact with an initial velocity of 40 ft/sec at an angle of 60° and slips by the opposing team untouched.

a. Find a vector equation for the path of the volleyball. (4 points)

$$\vec{r}(t) = (v_0 t\cos\theta)\hat{i} + \left(-\tfrac{1}{2}g t^2 + v_0 t\sin\theta + h_0\right)\hat{j}$$
$$= (40\cos 60°)t\,\hat{i} + (-16t^2 + 40\sin 60°\cdot t + 3)\hat{j}$$
$$20t\,\hat{i} + (-16t^2 + 20\sqrt{3}t + 3)\hat{j}$$

b. How high does the volleyball go? (4 points)

$$-32t + 20\sqrt{3} = 0$$
$$\frac{\cancel{32}t}{\cancel{32}} = \frac{20\sqrt{3}}{32} \Rightarrow t \approx 1.08$$

max y is ≈ 21.75 ft. (plug into $\hat{j}$ coord)

c. Find its range. (4 points)

t = 2.25

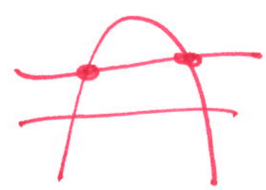
$$20(2.25) \approx 45 \text{ ft.}$$

plug into $\hat{i}$ coord.

d. When is the volleyball 10 feet above the ground? (4 points)

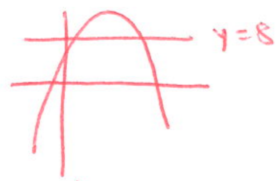
t = .226 seconds

t = 1.939 seconds



e. Suppose that the net is raised to 8 feet. Will the ball sail over the net? Explain by showing work. (4 points)

$t = .1556$



$20(.1556) = 3.1\ \hat{\imath}$

Yes, as long as the player is more than 3.1 feet away (which they are at 10 ft)

10. Find the principle unit normal vector to the curve $\vec{r}(t) = 2\sin t\,\hat{\imath} + 2\cos t\,\hat{\jmath}$, at the point $t = \frac{5\pi}{4}$. (10 points)

$\vec{r}\,'(t) = 2\cos t\,\hat{\imath} - 2\sin t\,\hat{\jmath}$ $\quad \vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} =$

$\dfrac{2\cos t\,\hat{\imath} - 2\sin t\,\hat{\jmath}}{2} = \cos t\,\hat{\imath} - \sin t\,\hat{\jmath}$

$\vec{T}'(t) = -\sin t\,\hat{\imath} - \cos t\,\hat{\jmath}$ $\quad \vec{N}(t) = \dfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$

$= -\sin t\,\hat{\imath} - \cos t\,\hat{\jmath}$

$\vec{N}\left(\frac{5\pi}{4}\right) = \frac{\sqrt{2}}{2}\hat{\imath} + \frac{\sqrt{2}}{2}\hat{\jmath}$

11. Find the length of the space curve $\vec{r}(t) = -3\cos t\,\hat{\imath} + 3\sin t\,\hat{\jmath} + t\hat{k}$ over the interval $[0,2\pi]$. (10 points)

$\vec{r}\,'(t) = 3\sin t\,\hat{\imath} + 3\cos t\,\hat{\jmath} + 1\,\hat{k}$

$\|\vec{r}\,'(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t + 1} = \sqrt{10}$

$\int_0^{2\pi} \|\vec{r}\,'(t)\|\,dt = \int_0^{2\pi} \sqrt{10}\,dt = \boxed{2\pi\sqrt{10}}$

12. Find the curvature of the curve given by $\vec{r}(t) = t\hat{\imath} + 2t^2\hat{\jmath} + 3t\hat{k}$. (10 points)

$$\vec{r}\,'(t) = 1\hat{\imath} + 4t\hat{\jmath} + 3\hat{k}$$

$$\vec{r}\,''(t) = 0\hat{\imath} + 4\hat{\jmath} + 0\hat{k}$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 4t & 3 \\ 0 & 4 & 0 \end{vmatrix} =$$

$$-12\hat{\imath} - 0\hat{\jmath} + 4\hat{k}$$

$$\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = \sqrt{(-12)^2 + 4^2} = \sqrt{144+16} = \sqrt{160} = 4\sqrt{10}$$

$$\|r'(t)\| = \sqrt{1^2 + 16t^2 + 9} = \sqrt{10 + 16t^2}$$

$$K = \frac{4\sqrt{10}}{(10+16t^2)^{3/2}}$$