MATH 240, Exam #4, Summer 2017 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Determine whether the vector field $\vec{F}(x,y,z) = \frac{1}{y}\hat{\imath} - \frac{x}{y^2}\hat{\jmath} + (3z^2 - z)\hat{k}$ is conservative. If it is, find a potential function for the vector field. (10 points)

$\int \frac{1}{y}\,dx = \frac{x}{y} + C_1(y,z)$

$\int -\frac{x}{y^2}\,dy = \frac{x}{y} + C_2(x,z)$

$\int 3z^2 - z\,dz = z^3 - \frac{1}{2}z^2 + C_3(x,y)$

$\varphi(x,y,z) = \frac{x}{y} + z^3 - \frac{1}{2}z^2 + K$

yes, The field is Conservative and a potential function exists

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{1}{y} & -\frac{x}{y^2} & 3z^2 - z \end{vmatrix} = (0-0)\hat{\imath} - (0-0)\hat{\jmath} + \left(-\frac{1}{y^2} + \frac{1}{y^2}\right)\hat{k} = \vec{0}$$

2. Evaluate the line integral $\int_C (x^2 + y^2 + z^2)ds$ along the path $C: \vec{r}(t) = \sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + 2t\hat{k}$. (15 points) $[0, 2\pi]$

$r'(t) = \cos t\,\hat{\imath} - \sin t\,\hat{\jmath} + 2\hat{k}$

$\|r'(t)\| = \sqrt{\cos^2 + \sin^2 t + 4} = \sqrt{5}\,dt$

$\int_0^{2\pi} (\sin^2 t + \cos^2 t + 4t^2)\sqrt{5}\,dt$

$= \sqrt{5}\int_0^{2\pi} 1 + 4t^2\,dt = \sqrt{5}\left[t + \frac{4}{3}t^3\right|_0^{2\pi} =$

$\sqrt{5}\left[2\pi + \frac{4}{3}(2\pi)^3\right] = \boxed{\sqrt{5}\left[2\pi + \frac{32\pi^3}{3}\right]}$

3. Find the work done by the force field $\vec{F}(x,y,z) = -\frac{1}{2}x\hat{\imath} - \frac{1}{2}y\hat{\jmath} + \frac{1}{4}\hat{k}$ on a particle as it moves along the helix given by $\vec{r}(t) = t\hat{\imath} + \sin 2t\,\hat{\jmath} + \cos 2t\,\hat{k}$ from the point $(0,0,1)$ to $(3\pi, 0, -1)$. (15 points)

$$\int -\tfrac{1}{2}x\,dx = \frac{-x^2}{4} + C_1(y,z)$$
$$\int -\tfrac{1}{2}y^2\,dy = -\frac{y^2}{4} + C_2(x,z)$$
$$\int \tfrac{1}{4}\,dz = \tfrac{1}{4}z + C_3(x,y)$$

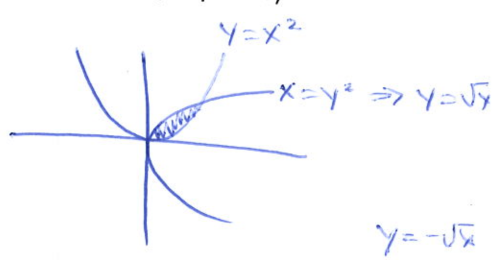
field is conservative

$$\varphi(x,y,z) = \frac{-x^2}{4} - \frac{y^2}{4} + \frac{1}{4}z + K$$

$$\int_C \vec{F}\cdot d\vec{r} = \varphi(3\pi,0,-1) - \varphi(0,0,1) =$$
$$-\frac{(3\pi)^2}{4} - 0 + \tfrac{1}{4}(-1) - \left(0^2 - 0^2 + \tfrac{1}{4}(1)\right) =$$
$$-\frac{9\pi^2}{4} - \frac{1}{4} - \frac{1}{4} = \boxed{-\frac{9\pi^2}{4} - \frac{1}{2}}$$

4. Use Green's Theorem $\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$ to evaluate $\oint_C [(x^3 - x^2y)dx + xy^2dy]$ where C is the boundary of the region R bounded by $y = x^2, x = y^2$. (16 points)

$$\frac{\partial M}{\partial y} = -x^2 \qquad \frac{\partial N}{\partial x} = y^2$$



$$\int_0^1 \int_{x^2}^{\sqrt{x}} y^2 + x^2\,dy\,dx =$$

$$\int_0^1 \tfrac{1}{3}y^3 + x^2y \Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 \tfrac{1}{3}x^{3/2} + x^{5/2} - \tfrac{1}{3}x^6 - x^4\,dx =$$

$$\tfrac{1}{3}\cdot\tfrac{2}{5}x^{5/2} + \tfrac{2}{7}x^{7/2} - \tfrac{1}{3}\cdot\tfrac{1}{7}x^7 - \tfrac{1}{5}x^5 \Big|_0^1 =$$

$$\frac{2}{15} + \frac{2}{7} - \frac{1}{21} - \frac{1}{5} = \boxed{\frac{6}{35}}$$

5. a. Find a parametrization of the portion of the plane $x + y + z = 2$ inside the cylinder $x^2 + y^2 = 1$. (10 points)

$z = 2 - v\cos u - v\sin u$

$x = v\cos u,\ y = v\sin u$

$u \in [0, 2\pi]$

$v \in [0, 1]$

$$\vec{r}(u,v) = (v\cos u)\hat{i} + (v\sin u)\hat{j} + (2 - v\cos u - v\sin u)\hat{k}$$

b. Use that parametrization to find the value of $\iint_R y\,dS$. (10 points)

$$\vec{r}_u = (-v\sin u)\hat{i} + (v\cos u)\hat{j} + (v\sin u - v\cos u)\hat{k}$$

$$\vec{r}_v = (\cos u)\hat{i} + (\sin u)\hat{j} + (-\cos u - \sin u)\hat{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -v\sin u & v\cos u & v\sin u - v\cos u \\ \cos u & \sin u & -\cos u - \sin u \end{vmatrix} =$$

$$(-v\cos^2 u - \cancel{v\cos u\sin u} - v\sin^2 u + \cancel{v\sin u\cos u})\hat{i} - (\cancel{v\sin u\cos u} + v\sin^2 u - \cancel{v\sin u\cos u} + v\cos^2 u)\hat{j} + (-v\sin^2 u - v\cos^2 u)\hat{k} =$$

$$(-v)\hat{i} - v\hat{j} - v\hat{k}$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{v^2 + v^2 + v^2} = \sqrt{3}\,v$$

$y = v\sin u$

$$\int_0^1 \int_0^{2\pi} \sqrt{3}\,v \cdot v\sin u \, du\, dv = \int_0^1 \sqrt{3}\,v^2 (-\cos u)\Big|_0^{2\pi} dv = 0$$

$-1 - (-1) = 0$

6. a. Find the divergence of $\vec{F} = x^2\hat{\imath} + 2y\hat{\jmath} + 4z^2\hat{k}$. (10 points)

$\vec{\nabla} \cdot \vec{F} = 2x + 2 + 8z$

b. Use the divergence Theorem $\iint_R \vec{F} \cdot \vec{n} dS = \iiint_V div \vec{F} dV$ to find the values of $\iint_R \vec{F} \cdot \vec{n} dS$ where S is the surface of the cylinder $x^2 + y^2 = 4, 0 \leq z \leq 2$ using the field above. (12 points)

$X = r\cos\theta = 2\cos\theta$
$Y = 2\sin\theta$

$\int_0^{2\pi} \int_0^2 \int_0^2 (2 \cdot 2\cos\theta + 2 + 8z) r \, dz \, dr \, d\theta =$

$\int_0^{2\pi} \int_0^2 4r\cos\theta z + 2rz + 4z^2 r \Big|_0^2 dr \, d\theta =$

$\int_0^{2\pi} \int_0^2 8r\cos\theta + 4r + 16r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 8r\cos\theta + 20r \, dr \, d\theta$

$= \int_0^{2\pi} 4r^2\cos\theta + 10r^2 \Big|_0^2 = \int_0^{2\pi} 16\cos\theta + 40 \, d\theta =$

$16\sin\theta + 40\theta \Big|_0^{2\pi} = 0 + 40(2\pi) - 0 - 0 =$

$\boxed{80\pi}$

7. a. Find the curl of $\vec{F} = x\hat{\imath} + (x+y)\hat{\jmath} + (x+y+z)\hat{k}$. (9 points)

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & x+y & x+y+z \end{vmatrix} = (1-0)\hat{\imath} - (1-0)\hat{\jmath} + (1-0)\hat{k} =$$

$$\langle 1,-1,1 \rangle$$

b. Find the parametric equation of the surface defined by $x = 2\cos t,\ y = 2\sin t,\ z = 2$, $0 \le t \le 2\pi$. (8 points)

$$\vec{r}(u,v) = v\cos u\,\hat{\imath} + v\sin u\,\hat{\jmath} + 2\hat{k} \quad u \in [0, 2\pi],\ v \in [0,2]$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 0 \end{vmatrix} = (0-0)\hat{\imath} - (0-0)\hat{\jmath} + (-v\sin^2 u - v\cos^2 u)\hat{k} =$$

$$= -0\,\hat{\imath} + 0\,\hat{\jmath} - v\hat{k}$$

c. Use that to evaluate $\oint_C \vec{F} \cdot d\vec{r} = \iint_R (curl\ \vec{F}) \cdot \vec{n}dS$ for the surface defined above. (8 points)

$$\vec{\nabla} \times \vec{F} \cdot \vec{r}_u \times \vec{r}_v = -v$$

$$\int_0^2 \int_0^{2\pi} -v\, du\, dv = \int_0^2 -vu \Big|_0^{2\pi} dv =$$

$$\int_0^2 -2\pi v + 0\, dv = \int_0^2 -2\pi v\, dv = -\pi v^2 \Big|_0^2 =$$

$$\boxed{-4\pi}$$