5/17/2021

Vectors

3d Coordinate Systems

The Righthand Rule, and The Lefthand Rule… in math we use the right-hand rule
X points out of the page, y is pointing to the right, and z is pointing up: label your axes

See handout on plotting points in 3D.

Visualizing equations is 3D.

In 2D: $x = 4$, $y = 2$, or $x + y = 7$ are all lines.
In 3D, we can't draw a line with just one equation: There is another direction available and it's free.
1 or 2 variable equations create not lines, but planes.
$Z = 3$ is a plane, parallel to the xy-plane but 3 units up.

Any curve with only two variables, the third variables is free to be any anything: get a sheet : cylinders

$$x^2 + y^2 = 1$$

Circle of radius 1 in the plane, but in 3D, this is a circular cylinder.

Distance formula for three dimensions:

$$dist(P_1, P_2) = |P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Find the distance between $(3, 5, 2)$ and $(-4, 1, -6)$.

$$d = \sqrt{\left(3 - (-4)\right)^2 + (5 - 1)^2 + \left(2 - (-6)\right)^2} = \sqrt{49 + 16 + 64} = \sqrt{129}$$

The equation of a sphere

Center (h, k, l), radius is ρ.

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = \rho^2$$

Suppose that $(3, 5, 2)$ and $(-4, 1, -6)$ are the ends of a diameter of a sphere. What is the equation of the sphere?

$$\rho = \frac{\sqrt{129}}{2}$$

Center is the midpoint of the diameter.

Center = $\left(\frac{3+(-4)}{2}, \frac{5+1}{2}, \frac{2+(-6)}{2}\right) = \left(-\frac{1}{2}, 3, -2\right)$

$$\left(x+\frac{1}{2}\right)^2 + (y-3)^2 + (z+2)^2 = \frac{129}{4}$$

Vectors are kinda like rays: they start at a point, and then point off in a particular direction: length and the direction determine a vector

Direction is θ, and distance r (magnitude, length)

Can also be in component form: $\langle a, b, c\rangle$ (angle bracket notation), (a, b, c), $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, unit vector format: $a\hat{\imath} + b\hat{\jmath} + c\hat{k}$.

$$\hat{\imath} = < 1,0,0 >$$
$$\hat{\jmath} = < 0,1,0 >$$
$$\hat{k} = < 0,0,1 >$$

$$\langle 3, 7, -2\rangle = 3\hat{\imath} + 7\hat{\jmath} - 2\hat{k}$$

Don't mix them.

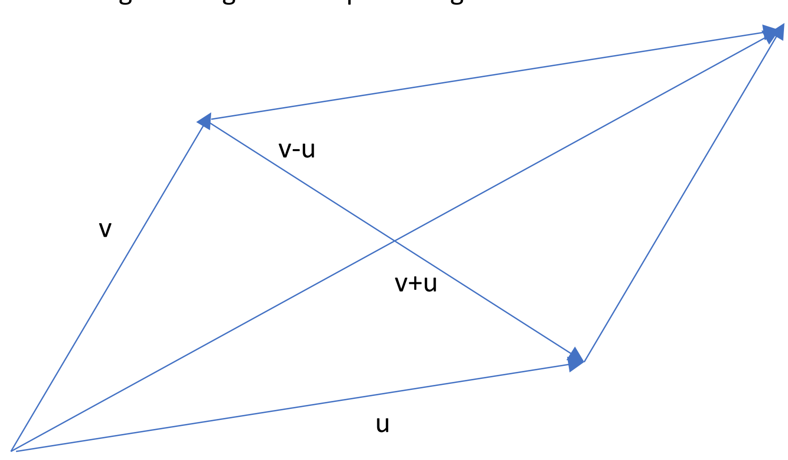
Combine vectors: add them, and rescale them.

Add vectors: add the corresponding components
Scaling vectors: scale all the components by the same factor

$$\vec{v} = \langle 3,7,-2\rangle, \vec{u} = \langle -1, 3, 5\rangle$$
$$\vec{v} + \vec{u} = \langle 2,10,3\rangle$$
$$\vec{v} - \vec{u} = \langle 4,4,-7\rangle$$

$$2\vec{v} = \langle 6,14,-4\rangle$$

Adding vectors is equivalent finding the diagonal of a parallelogram.



Magnitude of vector is the length of the vector, distance between the ending point, and the origin
Distance formula, $\vec{v} = \langle a, b, c \rangle$

$$r = |v| = \|\vec{v}\| = \sqrt{a^2 + b^2 + c^2}$$

Properties of Vectors:
Addition commutativity
Addition associativity
Distributive rule (for one scalar and two vectors, or two scalars and one vectors)
Associativity for multiplication of scalars $(cd)\vec{a} = c(d\vec{a})$
Identity of addition $\vec{a} + (-\vec{a}) = \vec{0}$
Scalar identity is 1

Convert a vector to a unit vector (how we define direction in 3D)

$$\frac{\vec{v}}{\|v\|} = \frac{\langle 3,7,-2 \rangle}{\sqrt{9 + 49 + 4}} = \langle \frac{3}{\sqrt{62}}, \frac{7}{\sqrt{62}}, -\frac{2}{\sqrt{62}} \rangle$$

$\langle \frac{3}{\sqrt{58}}, \frac{7}{\sqrt{58}} \rangle$ in 2D, first component would be $\cos(\theta)$, and the second component would be $\sin(\theta)$.

Sometimes direction cosines: unit vector $\langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$, the angles are the angles the vector makes with each coordinate axis.

For 2D: $r = \|\vec{v}\|$, and direction is θ.
In component form: $\langle r\cos(\theta), r\sin(\theta) \rangle$

Dot Product

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$
$$= \|a\|\|b\| \cos(\theta_{between})$$

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{\|a\|\|b\|}$$

If the vectors are perpendicular (at right angles to each other), then the dot product is 0.
If the angle between is acute, the dot product is positive
If the angle between is obtuse, the dot product will be negative

Projection formula
(orthogonal projection)

$$proj_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|}(\vec{a})$$

Orthogonal complement is the part of the vector perpendicular to the base vector:

$$comp_{\vec{a}}\vec{b} = \vec{b} - proj_{\vec{a}}\vec{b}$$

Cross product

$$\vec{a} \times \vec{b}$$

Produces a vector that is perpendicular to both of the original vectors.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 7 & -2 \\ -1 & 3 & 5 \end{vmatrix} = \big(7(5) - (3)(-2)\big)\hat{\imath} - \big(3(5) - (-1)(-2)\big)\hat{\jmath} + \big(3(3) - (-1)(7)\big)\hat{k} =$$

$$41\hat{\imath} - 13\hat{\jmath} + 16\hat{k}$$

$$41(3) - 13(7) + 16(-2) = 0$$

Geometry dot product and the cross product

The magnitude of the cross product is the area of the parallelogram defined by the two vectors.

$$\sqrt{41^2 + (-13)^2 + 16^2} = \sqrt{2106}$$

If I have three vectors $\left|(\vec{a} \times \vec{b}) \cdot \vec{c}\right|$ is called the triple scalar product is the volume of a parallelepiped defined by the three vectors.

$$\left|(\vec{a} \times \vec{b}) \cdot \vec{c}\right| = \begin{vmatrix} c_1 & c_2 & c_3 \\ 3 & 7 & -2 \\ -1 & 3 & 5 \end{vmatrix}$$

Triangles: half the parallelogram

Pyramid: divide by six