5/27/2021

Iterated/Double Integrals (15.1-15.3)

Potential functions, we were using integrals to perform anti-derivatives and the integrals were indefinite integrals. Potential functions are the only place where we have indefinite integrals in multiple variables.

ie. $\int f(x,y,z)dx$

When we combine integrals (two, three, etc.), they are always definite integrals: they always have limits of integration.

ie. $\int_a^b \int_{h(x)}^{g(x)} f(x,y)dA$

dA is a shorthand for (in rectangular coordinates) $dxdy$ or $dydx$. When we integrate the "inside" variable (the one on the left) is the one that gets integrate first. This is why its sometimes called an iterated integral, because we integrate in one variable at a time, and then the result is pushed into the next integral for the next variable… until we run out of variables.

Remember when we did our 1-variable integrals, the dx stood for the small width of the rectangle as we divided up our base for the area calculation. The base here is in two-dimensions, as a rectangle, and the distance of that base in the x-direction is dx, and the base in the y-direction is dy. The volume is calculated by multiplying the dy and dx and the height of the function.

You can do estimations of volume this way, by dividing up the plane into small rectangles, and then estimating the volume of the region at some point above the rectangle, and then adding up the little boxes.

Another simplification you may see in notation:

$$\iint\limits_R f(x,y)dA$$

R stands for "region" in the plane that the integral is being taken over.

The limits of integration. There are some rules:

1) The outer limits of integration are expected to be a constant. So if the height function is a function of x and y, the limits of integration cannot contain x or y. When you get to the last integral (where these limits apply) the problem should look exactly like a single integral from Calc I.
2) Inner limits of integration, they cannot contain any variables that are being integrated on that step, or that have been previous integrated. So if you are integrating y first (in the two-variable care), then the limits of integration can be functions of x. If you are integrating x first, then then the limits of integration can be functions of y.

Examples.

$$\int_{-3}^{3}\int_{0}^{\frac{\pi}{2}}(y + y^2\cos x)dxdy$$

Think about this like:

$$\int_{-3}^{3}\left[\int_{0}^{\frac{\pi}{2}}(y + y^2\cos x)dx\right]dy$$

$$\int_{-3}^{3}[yx + y^2\sin x]_{x=0}^{x=\pi/2}\,dy$$

$$\int_{-3}^{3}\frac{\pi}{2}y + y^2 dy$$

$$\frac{\pi}{2}\cdot\frac{1}{2}y^2 + \frac{1}{3}y^3\Big|_{-3}^{3} = \frac{9\pi}{4} + \frac{1}{3}(27) - \left(\frac{9\pi}{4} + \frac{1}{3}(-27)\right) = 9 + 9 = 18$$

$$\iint_R \frac{xy^2}{x^2+1}dA\,, R = \{(x,y)|0 \le x \le 1, -3 \le y \le 3\}$$

Or R: $[0,1] \times [-3,3]$

$$\int_{0}^{1}\int_{-3}^{3}\frac{xy^2}{x^2+1}dy\,dx$$
$$\int_{0}^{1}\frac{x}{x^2+1}\int_{-3}^{3}y^2 dy\,dx$$

$$\int_{0}^{1}\frac{x}{x^2+1}\left[\frac{1}{3}y^3\right]_{-3}^{3}dx$$

$$\int_{0}^{1}\frac{x}{x^2+1}(18)dx = 18\cdot\frac{1}{2}\ln|x^2+1|\Big|_{0}^{1} = 9(\ln(2)) = 9\ln 2$$

Here so far, our limits are all constant. And we are free to switch the order of integration as we see fit. The answer will be the same either way. The region in the plane that we are integrating over is a rectangle.

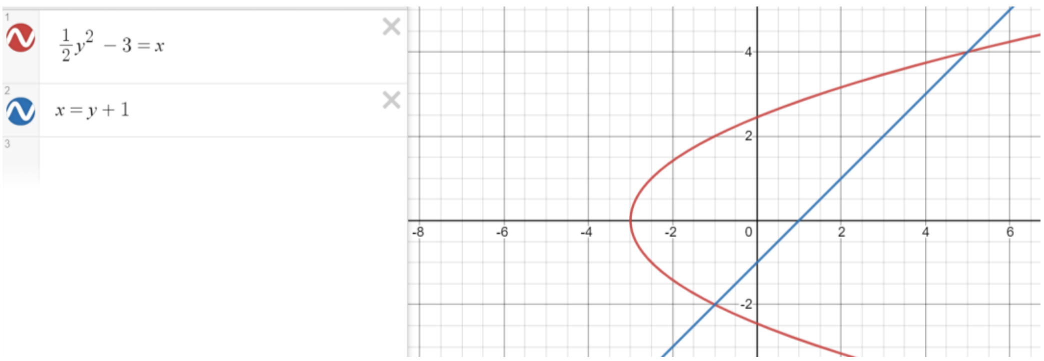
Look at general regions rather than just rectangular regions.
Doing area in double integrals.

$$\iint_R dA$$

Region:

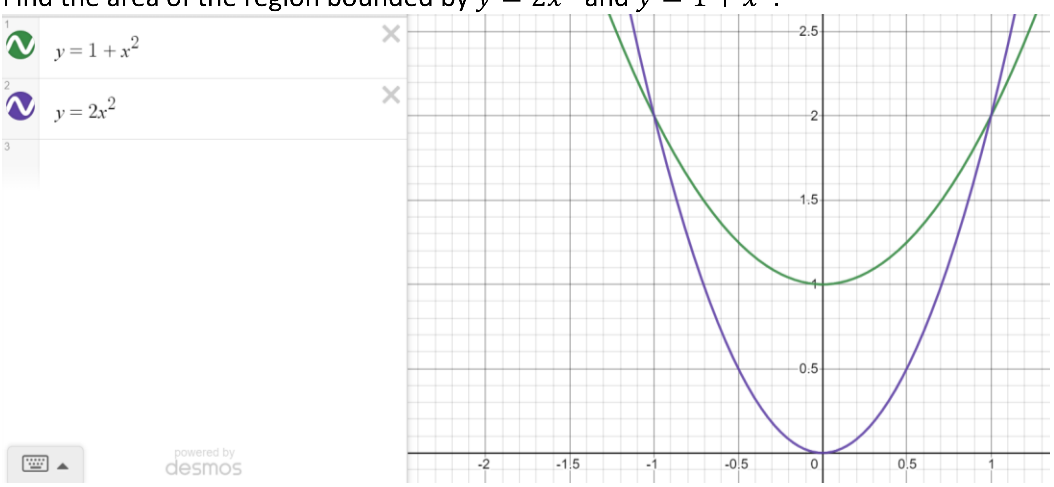
$$\left\{(x,y)\middle|-2 \le y \le 4, \frac{1}{2}y^2 - 3 \le x \le y+1\right\}$$

Find the area of the region bounded by the curves $x = \frac{1}{2}y^2 - 3$ and $x = y + 1$.



$$\int_{-2}^{4} f(y) - g(y)dy = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} 1dx\,dy = \int_{-2}^{4} (y+1) - (\frac{1}{2}y^2 - 3)\,dy$$

Find the area of the region bounded by $y = 2x^2$ and $y = 1 + x^2$.



$$\int_{-1}^{1} f_{top}(x) - f_{bottom}(x)\,dx = \int_{-1}^{1}\int_{2x^2}^{1+x^2} 1dy\,dx = \int_{-1}^{1} (1 + x^2) - (2x^2)dx$$

For volume we swap out the 1 for area with a function that describes the height of the surface in terms of x and y.

Volume:

Find the volume under the surface $f(x, y) = x^2 + 2y$ over the region bounded by $y = x, y = x^3, x \ge 0$.

$$\int_0^1 \int_{x^3}^{x} x^2 + 2y dy dx = \int_0^1 x^2 y + y^2 |_{y=x^3}^{y=x} dx = \int_0^1 x^3 + x^2 - (x^5 + x^6) dx$$

$$= \int_0^1 x^3 + x^2 - x^5 - x^6 dx = \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{6}x^6 - \frac{1}{7}x^7 \Big|_0^1 = \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{7} = \frac{23}{84}$$

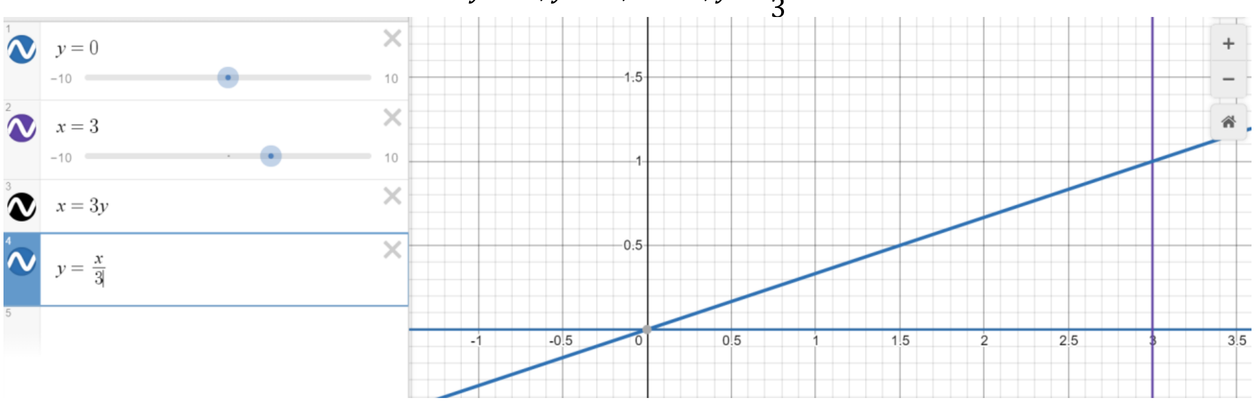
What happens if set the integral and we can't actually integrate it?

$$\int_0^1 \int_{3y}^{3} e^{x^2} dx dy$$

We can't integrate this as is, but if we change the order of integration, we can.

$$y = 0, y = 1, x = 3, x = 3y$$

$$y = 0, y = 1, x = 3, y = \frac{x}{3}$$



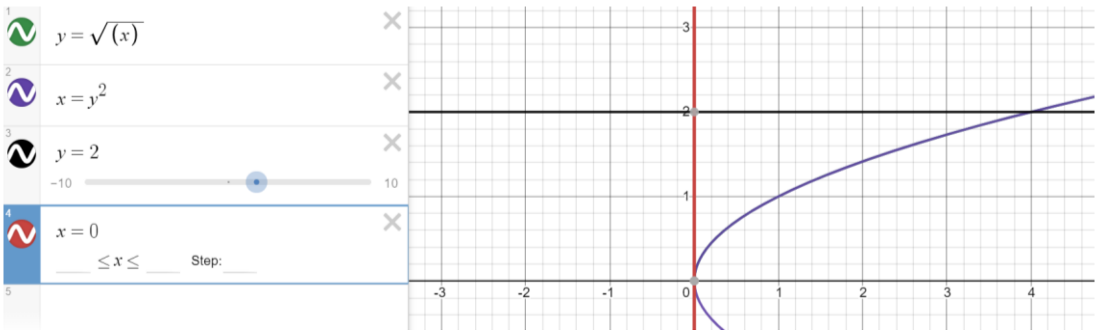
$$\int_0^3 \int_0^{x/3} e^{x^2} dy dx = \int_0^3 y e^{x^2} \Big|_0^{x/3} dx = \frac{1}{3} \int_0^3 x e^{x^2} dx = \frac{1}{3} \cdot \frac{1}{2} e^{x^2} \Big|_0^3 = \frac{1}{6} e^9 - \frac{1}{6}$$

Example.

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1} dydx$$

$$y = \sqrt{x}, y = 2, x = 0, x = 4$$

$$x = y^2, y = 2, x = 0, x = 4$$



$$\int_0^2 \int_0^{y^2} \frac{1}{y^3+1} dxdy = \int_0^2 \frac{x}{y^3+1}\bigg|_0^{y^2} dy = \int_0^2 \frac{y^2}{y^3+1} dy = \frac{1}{3}\ln(y^3+1)\bigg|_0^2 = \frac{1}{3}\ln(9)$$