6/10/2021

Surface Area (Parametric) (16.6)
Line Integral Theorems (16.3,16.4)

Last time we established that the surface area is the integral of the magnitude of the vector normal to the tangent plane. ∇F is the normal to the tangent plane in rectangular coordinates (move everything to one side, set =0, that is F, and then find the gradient of that), and the normal to the tangent plane in parametric form is also used for surface area.

$$A(S) = \iint_R \|\vec{N}\| dA = \iint_R \|\nabla F\| dA = \iint_R \sqrt{f_x^2 + f_y^2 + 1} dA = \iint_R \|\vec{r}_u \times \vec{r}_v\| dA$$

Example.

$$r(u, v) = \langle u \cos v \, , u \sin v \, , v \rangle$$

$$u \in [0,1], v \in [0, \pi]$$

Find the surface area.

$$r_u = \langle \cos v \, , \sin v \, , 0 \rangle$$
$$r_v = \langle -u \sin v \, , u \cos v \, , 1 \rangle$$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{vmatrix} = (\sin v - 0)i - (\cos v - 0)j + (u \cos^2 v + u \sin^2 v)k =$$
$$\langle \sin v \, , -\cos v \, , u \rangle$$

$$\|r_u \times r_v\| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1 + u^2}$$

$$A(S) = \int_0^1 \int_0^\pi \sqrt{1 + u^2} dv du = \int_0^1 \pi \sqrt{1 + u^2} du$$

Finish using trig substitution.

Line Integral Theorems:
1) The Fundamental Theorem for Line Integrals (fields with special properties)
2) Green's Theorem (curves with special properties)

The Fundamental Theorem of Line Integrals is essentially a theorem that is similar to the Fundamental Theorem of Calculus, in that it relates an antiderivative evaluated at the endpoints to the result of integration (line integral).

For the Fundamental Theorem of Line Integrals to apply, the field we are integrating over must be conservative. (two things are true: 1) the curl is zero, 2) there is a potential function that can be used to derive the field, ie. $f(x, y, z)$ exists so that $F(x, y, z) = \nabla f(x, y, z)$).

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

Find the potential function and evaluate the potential function at the starting point and ending point in space.

In a conservative vector field, the line integral is independent of path: if you start and stop at the same points, the line integral will have the same value regardless of how you get there.

If the problem tells you that you should apply the Fundamental Theorem of Line Integrals should may assume that the field IS conservative: can go directly to finding the potential. If the problem says you "may" use the Fundamental Theorem of Line Integrals, "if it applies", you need to check that the field is conservative.

Example.

$$F(x, y, z) = \langle yz, xz, xy + 2z\rangle$$

C is the line segment from $(1,0,-2)$ to $(4,6,3)$.

(Old way: find the parametric form of the line, find the derivative of the line, dot with the vector field, and then integrate from 0 to 1 in t)

New way: find the potential function, and then plug in the points in space and subtract.

$$\int yzdx = xyz + stuff$$

$$\int xzdy = xyz + stuff$$

$$\int xy + 2zdz = xyz + z^2 + stuff$$

$$f(x, y, z) = xyz + z^2$$

$$\int_0^1 \langle yz, xz, xy + 2z\rangle \cdot r'(t)dt = f(4,6,3) - f(1,0,-2) = 72 + 9 - (0 + 4) = 77$$

Example.

$$\int_C 2xe^{-y}dx + (2y - x^2e^{-y})dy$$

C is any path from (1,0) to (2,1)

$$F(x, y) = \langle 2xe^{-y}, 2y - x^2e^{-y}\rangle$$

$$F(x, y) \cdot r(t)dt = 2xe^{-y}(x'(t))dt + (2y - x^2e^{-y})(y'(t))dt$$

Find the potential for the field and apply the Fundamental Theorem of Line Integrals.

$$\int 2xe^{-y}dx = x^2e^{-y} + stuff$$

$$\int 2y - x^2e^{-y}dy = y^2 + x^2e^{-y} + stuff$$

$$f(x,y) = x^2e^{-y} + y^2$$

$$\int_C 2xe^{-y}dx + (2y - x^2e^{-y})dy = f(2,1) - f(1,0) = 4e^{-1} + 1 - (1(1) - 0) = \frac{4}{e}$$

The Fundamental Theorem of Line Integrals applies to conservative vector fields only: only situation where potential function exists, and integral result is independent of path.

Green's Theorem: applies only to situtations where the curve is closed—starts and stops at the same place.
Generally does not apply when the field is conservative, because if you are starting and stopping in the same place, and the field is conservative, then the integral is zero, always zero.

It turns out that for closed curves in a non-conservative field, the value of the line integral is related to the area enclosed by the curve.

Green's Theorem is the 2D version of a more generally 3D theorem (which we'll maybe look at later). And it's usually expressed as the differential form of the line integral.

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Example.

$$\int_C xydx + x^2dy$$

C is the rectangle with vertices (0,0), (3,0), (3,1), (0,1).

(old way: parametrize each segment of the rectangle, calculate the line integral on each of the 4 segments, add up to get the result.)

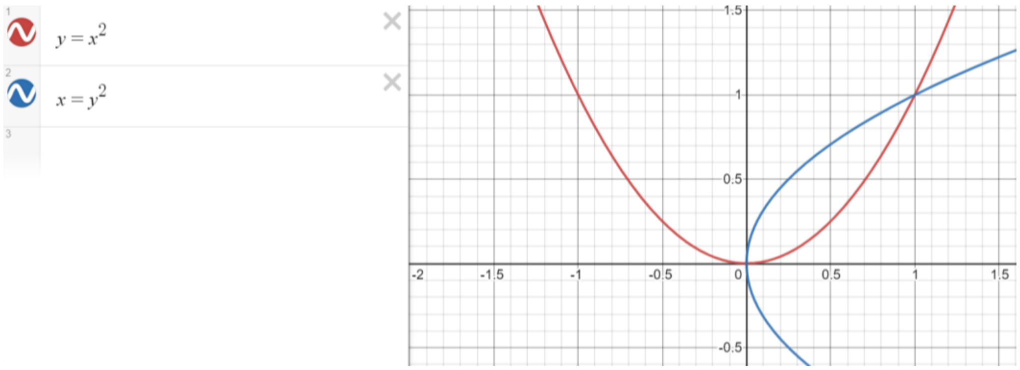
New way: apply Green's Theorem: take a derivative, set up an area integral, and solve.

$$\int_C xydx + x^2dy = \int_0^3 \int_0^1 (2x - x)dydx = \int_0^3 \int_0^1 x\,dydx = \int_0^3 xdx = \frac{1}{2}x^2\Big|_0^3 = \frac{9}{2}$$

Example.

$$F(x,y) = \langle y + e^{\sqrt{x}}, 2x + \cos y^2 \rangle$$

C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$

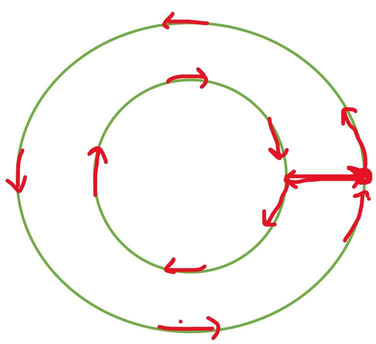


$$\int_C F \cdot dr = \int_0^1 \int_{x^2}^{\sqrt{x}} (2-1)\,dy\,dx$$

Here, the line integral is exactly equal to the area of the region.

You need to care about the orientation of the curve. Pay attention to problems (with ellipses and circles in particular) about whether they are going clockwise (weird) or counterclockwise (normal).

Green's Theorem can also be applied to regions that have holes in them, for instance, an annulus.



If you have a curve with 2 "separate" boundaries of the region, think about the curve where the curve starts and stops at the same place, it can revisit locations if needed, you may need to create "connecting" segments between the boundaries, that is traversed in the opposite direction so that it cancels out, and keep the area of the region always on the left.

Example.

$$\int_C (1-y^3)dx + \left(x^3 + e^{y^2}\right)dy$$

C is the boundary of the region between the circles $x^2 + y^2 = 4$, and $x^2 + y^2 = 9$.

$$\iint_R (3x^2 - (-3y^2))\,dA = 3\iint_R x^2 + y^2 dA = 3\int_0^{2\pi}\int_2^3 r^2 r\,dr\,d\theta = 3\int_0^{2\pi}\int_2^3 r^3\,dr\,d\theta$$

$$3(2\pi)[\frac{1}{4}r^4\Big|_2^3 = \frac{3\pi}{2}(81-16) = \frac{195\pi}{2}$$

If the integrand is a constant, you can shortcut the integration by using geometry formulas.

There are some problems in the book that are designed to this: the area of an ellipse is $A = \pi ab$ where a is the length of the semi-major axis, and b is the length of the semi-minor axis. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Summarize:

1) Old way is the brute force way. In theory, **this always works**, but it can be tedious, and in some cases you can't integrate by hand (depends on the field).
2) Fundamental Theorem of Line Integrals required conservative vector fields, and then you don't care about the path. And you may not get one.
3) Green Theorem depends on a closed curve, the path may be described as the boundary of a region (with or without an orientation—assume counterclockwise if not specified).