6/16/2021

Stokes' Theorem continued (16.8)

Theorem: $\int_C F \cdot dr = \iint_S (\nabla \times F) \cdot d\vec{S}$

$$F(x,y,z) = \langle 1, x+yz, xy-\sqrt{z}\rangle$$

C is the boundary of the part of the plane $3x+2y+z=1$ in the first octant.

$$G = 3x+2y+z-1 \rightarrow \nabla G = \langle 3,2,1\rangle$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & x+yz & xy-\sqrt{z} \end{vmatrix} = (x-y)i-(y-0)j+(1-0)k = \langle x-y,-y,1\rangle$$

$$\iint_S \langle x-y,-y,1\rangle \cdot \langle 3,2,1\rangle dA = \int_0^{1/3}\int_0^{(1-3x)/2} 3x-5y+1\, dydx$$

$$= \int_0^{\frac{1}{3}} 3xy-\frac{5}{2}y^2+y\Big|_0^{\frac{1-3x}{2}} dx = \int_0^{\frac{1}{3}} \frac{3x(1-3x)}{2} - \frac{\frac{5}{2}(1-3x)^2}{4} + \frac{1-3x}{2} dx =$$

$$\int_0^{\frac{1}{3}} \frac{3}{2}x-\frac{9}{2}x^2-\frac{5}{8}(1-6x+9x^2)+\frac{1}{2}-\frac{3}{2}xdx = \int_0^{\frac{1}{3}} -\frac{9}{2}x^2-\frac{5}{8}+\frac{15}{4}x-\frac{45}{8}x^2+\frac{1}{2}dx =$$

$$\int_0^{\frac{1}{3}} -\frac{81}{8}x^2+\frac{15}{4}x-\frac{1}{8}dx = -\frac{9}{8}x^3+\frac{15}{8}x^2-\frac{1}{8}x\Big|_0^{1/3} = -\frac{1}{24}+\frac{5}{24}-\frac{1}{24}=\frac{3}{24}=\frac{1}{8}$$

Example.

$$F(x,y,z) = \langle xy, 2z, 3y\rangle$$

C is the curve of intersection of the plane $x+z=5$ and the cylinder $x^2+y^2=9$.

$(r(t) = \langle 3\cos t, 3\sin t, 5-3\cos t\rangle)$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2z & 3y \end{vmatrix} = (3-2)i-(0-0)j+(0-x)k = \langle 1,0,-x\rangle$$

$$n = \langle 1,0,1\rangle$$

$$\iint_S \langle 1,0,-x\rangle \cdot \langle 1,0,1\rangle dA = \iint_S 1+0-xdA = \int_0^{2\pi}\int_0^3 (1-r\cos\theta)rdrd\theta$$

$$\int_0^{2\pi}\int_0^3 r - r^2\cos\theta\, drd\theta = \int_0^{2\pi} \frac{1}{2}r^2 - \frac{1}{3}r^3\cos\theta\Big|_0^3 d\theta = \int_0^{2\pi}\frac{9}{2} - 9\cos\theta\, d\theta$$

$$= \frac{9}{2}\theta + 9\sin\theta\Big|_0^{2\pi} = 9\pi$$

Exam #3 ends here.