6/22/2021

Extrema, Absolute Extrema (14.7)

Second Derivative Test for Critical Points (D-Test)
D = determinant
D is going to depend on the value of the second derivatives at the critical point

$$D = f_{xx}f_{yy} - f_{xy}^2$$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}f_{yx} = f_{xx}f_{yy} - f_{xy}^2$$

Example. Find and characterize/classify any critical points.

$$f(x,y) = x^2 + xy + y^2 + y$$

$$\nabla f = \langle 2x + y, x + 2y + 1 \rangle$$

Gradient is zero at a critical point.

$$f_x = 2x + y = 0$$
$$f_y = x + 2y + 1 = 0$$

$$y = -2x$$
$$x + 2(-2x) + 1 = 0$$
$$x - 4x = -1$$
$$-3x = -1$$
$$x = \frac{1}{3}$$
$$y = -\frac{2}{3}$$

Critical point $\left(\frac{1}{3}, -\frac{2}{3}\right)$.
Classify the critical point: use the 2nd derivative test.

$$f_{xx} = 2$$
$$f_{yy} = 2$$
$$f_{xy} = 1$$
$$D = f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - (1)^2 = 4 - 1 = 3$$

$D > 0$: relative extrema (either a max or a min)
$D < 0$: a saddle point
$D = 0$: undetermined

If we are in the extrema situation (max or a min), then to determine which one it is, look at the sign of f_{xx} (or f_{yy}). In the situation where $D > 0$, f_{xx} and f_{yy} with both have the same sign. If $f_{xx} > 0$, then the critical is a minimum (concave up in both directions). If $f_{xx} < 0$, then the critical is a maximum (concave down in both directions).

Example.

$$f(x,y) = xy(1-x-y) = xy - x^2y - xy^2$$

$$\nabla f = \langle y - 2xy - y^2, x - x^2 - 2xy \rangle$$

$$f_x = y - 2xy - y^2 = 0$$
$$f_y = x - x^2 - 2xy = 0$$

$$y(1-2x-y) = 0$$
$$x(1-x-2y) = 0$$

Critical point #1: x=0, y=0 or $(0,0)$

Critical point #2: y=0, $1 - x - 2(0) = 0 \to 1 - x = 0 \to x = 1$ or $(1,0)$

Critical point #3: x=0, $1 - 2(0) - y = 0 \to 1 - y = 0 \to y = 1$ or $(0,1)$

Critical point #4: $1 - 2x - y = 0, 1 - x - 2y = 0 \to y = 1 - 2x \to 1 - x - 2(1-2x) = 0$
$1 - x - 2 + 4x = 0 \to 3x - 1 = 0 \to 3x = 1 \to x = \frac{1}{3}, y = 1 - 2\left(\frac{1}{3}\right) = \frac{1}{3}$ or $\left(\frac{1}{3}, \frac{1}{3}\right)$

$$f_{xx} = -2y$$
$$f_{yy} = -2x$$
$$f_{xy} = 1 - 2x - 2y$$

Critical point (0,0):

$$f_{xx}(0,0) = 0, f_{yy}(0,0) = 0, f_{xy}(0,0) = 1$$
$$D = (0)(0) - (1)^2 = -1$$

This is a saddle point.

Critical point (1,0):

$$f_{xx}(1,0) = 0, f_{yy}(1,0) = -2, f_{xy}(1,0) = 1 - 2(1) - 0 = -1$$
$$D = (0)(-2) - (-1)^2 = -1$$

This is a saddle point.

Critical point (0,1):

$$f_{xx}(0,1) = -2, f_{yy}(0,1) = 0, f_{xy}(0,1) = 1 - 0 - 2(1) = -1$$
$$D = (-2)(0) - (-1)^2 = -1$$

This is a saddle point.

Critical point $\left(\frac{1}{3}, \frac{1}{3}\right)$:

$$f_{xx}\left(\frac{1}{3}, \frac{1}{3}\right) = -\frac{2}{3}, f_{yy}\left(\frac{1}{3}, \frac{1}{3}\right) = -\frac{2}{3}, f_{xy}\left(\frac{1}{3}, \frac{1}{3}\right) = 1 - 2\left(\frac{1}{3}\right) - 2\left(\frac{1}{3}\right) = 1 - \frac{2}{3} - \frac{2}{3} = 1 - \frac{4}{3} = -\frac{1}{3}$$

$$D = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) - \left(-\frac{1}{3}\right)^2 = \frac{4}{9} - \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$$

This is positive, so it's either a max or a min. And $f_{xx} < 0$ (is negative), so the graph is concave down, so this point is a maximum.

If you are dealing with trig functions, consider any critical points in the range of 0 to 2π, or you could use $-\pi$ to π.

Absolute Extrema.
In one variable functions, the basic steps were to find any critical points, check that they were contained in the interval of interest (or throw them out), then check the values of the critical points (that remain), and the endpoints of the interval. The largest value is the absolute max on the interval, and the smallest is the absolute minimum.

In two-variable problems (or higher), our basic steps will be to first the 2D critical points, then using the boundary functions for the region, subsitute into the 2D function to make one-variable functions and find the critical points on the boundaries, and find the corner points (are where the boundaries intersect with each other). Plug each point into the original function, and the largest value is the absolute maximum, and the smallest value is the absolute minimum.

Example. Find the absolute extrema on the function for the given region.

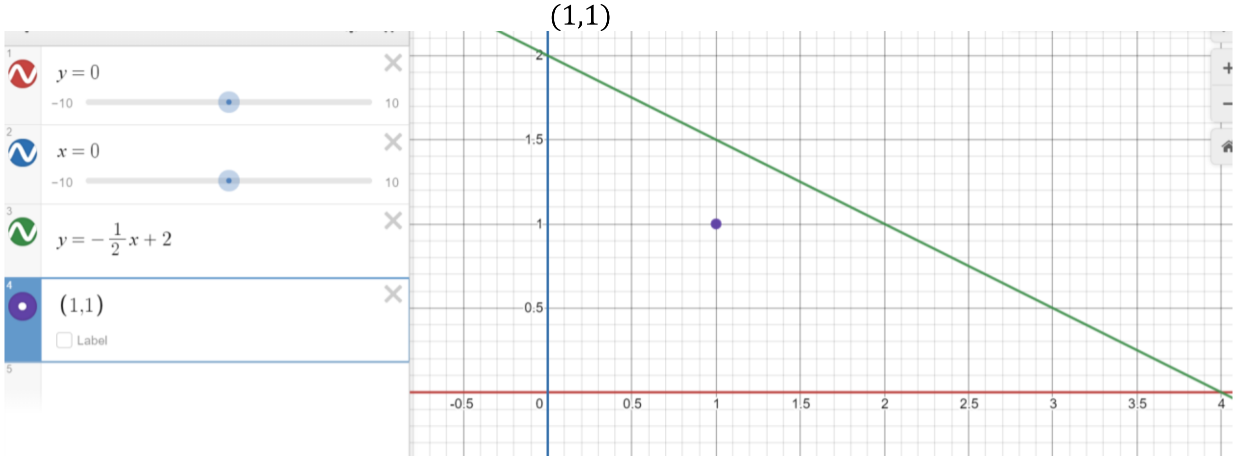
$$f(x,y) = x + y - xy$$

Region: the closed triangular region between the vertices $(0,0), (0,2), (4,0)$.
Functions that define the boundaries:

$$x = 0, y = 0, m = \frac{2-0}{0-4} = -\frac{1}{2}, y = -\frac{1}{2}x + 2$$

1) Find the relative extrema and see if they are in the region.

$$f_x = 1 - y = 0 \rightarrow y = 1$$
$$f_y = 1 - x = 0 \rightarrow x = 1$$
$$(1,1)$$



This is inside the region, so we keep it.

2) Check for critical points on the boundaries.
   a. Boundary $x = 0, f(x,y) = x + y - xy$

$$f(0,y) = y$$
$$f'(y) = 1$$

   This is never zero, so there are no critical points on this boundary.

b. Boundary $y = 0, f(x,y) = x + y - xy$

$$f(x,0) = x$$
$$f'(x) = 1$$

This is never zero, so there are no critical points on this boundary.

c. Boundary $y = -\frac{1}{2}x + 2, f(x,y) = x + y - xy$

$$f\left(x, -\frac{1}{2}x + 2\right) = x + \left(-\frac{1}{2}x + 2\right) - x\left(-\frac{1}{2}x + 2\right) = x - \frac{1}{2}x + 2 + \frac{1}{2}x^2 - 2x =$$
$$f(x) = 2 - \frac{3}{2}x + \frac{1}{2}x^2$$
$$f'(x) = -\frac{3}{2} + x = 0$$
$$x = \frac{3}{2}$$
$$y = -\frac{1}{2}\left(\frac{3}{2}\right) + 2 = -\frac{3}{4} + 2 = \frac{5}{4}$$
$$\left(\frac{3}{2}, \frac{5}{4}\right)$$

3) Check the cornerpoint. (If you started with equations, set the boundaries equal to each other in pairs—use the graph to find the intersections).
Check (0,0), (0,2), (4,0)

Critical points to check:

$$f(1,1) = 1 + 1 - 1(1) = 1$$

$$f\left(\frac{3}{2}, \frac{5}{4}\right) = \left(\frac{3}{2}\right) + \frac{5}{4} - \left(\frac{3}{2}\right)\left(\frac{5}{4}\right) = \frac{7}{8}$$

$$f(0,0) = 0 + 0 - 0(0) = 0$$
$$f(0,2) = 0 + 2 - 0(2) = 2$$
$$f(4,0) = 4 + 0 - 4(0) = 4$$

The absolute maximum is 4, at (4,0).
The absolute minimum is 0 at (0,0).

Example. Find the absolute extrema.

$$f(x,y) = xy^2$$

Region: $x \geq 0, y \geq 0, x^2 + y^2 \leq 3$

1) Find critical points.

$$f_x = y^2 = 0 \rightarrow y = 0$$
$$f_y = 2xy \rightarrow x = 0, y = 0$$

One critical point is (0,0). If y=0, then entire line is a critical value (a line of critical points)

2) Check the boundary conditions.

a. Boundary $x = 0$

$$f(0,y) = 0$$

$$f'(y) = 0$$

The entire line is a constant on this function (where the function value is 0).

b. Boundary $y = 0$

$$f(x, 0) = 0$$
$$f'(x) = 0$$

The entire line is a constant where the function value is 0.

c. Boundary $x^2 + y^2 = 3 \rightarrow y^2 = 3 - x^2$

$$f(x) = x(3 - x^2) = 3x - x^3$$
$$f'(x) = 3 - 3x^2 = 0$$
$$3 = 3x^2$$
$$1 = x^2$$

$x = \pm 1$ $\rightarrow$this is in the first quadrant only, so $x = 1$,$y^2 = 3 - (1)^2 = 2 \rightarrow y = \sqrt{2}$
$(1, \sqrt{2})$ (only positive since in quadrant 1)

3) Check the cornerpoints.
   a. $x = 0, y = 0 \rightarrow (0,0)$
   b. $x = 0, x^2 + y^2 = 3 \rightarrow y = \sqrt{3}, (0, \sqrt{3})$ – first quadrant positive
   c. $y = 0, x^2 + y^2 = 0 \rightarrow x^2 = 3 \rightarrow x = \sqrt{3}, (\sqrt{3}, 0)$ – first quadrant positive

Critical points to check:

$$f(0,0) = 0$$
$$f(1, \sqrt{2}) = 1(2) = 2$$
$$f(0, \sqrt{3}) = 0$$
$$f(\sqrt{3}, 0) = 0$$

Absolute extrema:
Absolute maximum is 2 at $(1, \sqrt{2})$
Absolute minima are 0 at $x = 0$ or $y = 0$.

Velocity and Acceleration (13.4)

Applying Newton's laws of motion
If you have an acceleration, the antiderivative is the velocity (and initial conditions will find the constants of integration for you), and the antiderivative of velocity will get you the position (and initial conditions will find the constants of integration here).

Using vector-valued functions.

$$a(t) = \langle 2, 6t, 12t^2 \rangle, v(0) = \langle 1,0,0 \rangle, r(0) = \langle 0,1,-1 \rangle$$

$$\int a(t)dt = v(t)$$

$$\int \langle 2, 6t, 12t^2 \rangle dt = \langle \int 2dt, \int 6tdt, \int 12t^2 dt \rangle = \langle 2t + C_1, 3t^2 + C_2, 4t^3 + C_3 \rangle = v(t)$$

$$v(0) = \langle 0 + C_1, 0 + C_2, 0 + C_3 \rangle = \langle 1,0,0 \rangle$$
$$C_1 = 1, C_2 = 0, C_3 = 0$$
$$v(t) = \langle 2t + 1, 3t^2, 4t^3 \rangle$$

$$\int v(t) = r(t)$$
$$\int \langle 2t + 1, 3t^2, 4t^3 \rangle dt = \langle \int 2t + 1dt, \int 3t^2 dt, \int 4t^3 dt \rangle = \langle t^2 + t + C_4, t^3 + C_5, t^4 + C_6 \rangle = r(t)$$
$$r(0) = \langle 0 + 0 + C_4, 0 + C_5, 0 + C_6 \rangle = \langle 0,1,-1 \rangle$$
$$C_4 = 0, C_5 = 1, C_6 = -1$$

$$r(t) = \langle t^2 + t, t^3 + 1, t^4 - 1 \rangle$$

Projectile motion:

$$r(t) = \langle (v_0 \cos\alpha)t, h_0 + (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \rangle$$

When solving these problems: they are designed to use 9.8 $\frac{m}{s^2}$ or 32 $\frac{ft}{s^2}$.

Suppose a baseball is hit from a height of 1 m off the ground at an angle of 30°, at a velocity of 50 m/sec. What is the range of the ball (where does it land), and is it a homerun if the height of the outfield fence is 4 m at a distance of 120 m from homeplate.

$$v_0 = 50$$
$$h_0 = 1$$
$$g = 9.8$$
$$\alpha = 30°$$

$$r(t) = \langle (50 \cos 30°)t, 1 + (50 \sin 30°)t - 4.9t^2 \rangle$$

Set the y-component equal to 0 to solve for t. Here you'll get a positive and a negative time value. Ignore the negtive time. The positive time value is when the ball hits the ground. To get the range, use that positive time value in the x-component to find the distance. To find how high the ball is when it gets to the outfield fence, set the x-component equal to the distance to the fence, and find t. Plug into the y-component to see how high it is when it gets to the fence line. If it's taller than the fence, it's a homerun. If it's shorter than the fence, then it hits the wall, and is not a homerun.