6/23/2021

Change of Variable (Jacobians) (15.10)
Centers of Mass (15.5, 15.7-15.9)

Jacobian is essentially a matrix of partial derivatives related to the way that our coordinate system is being changed: from rectangular to polar, or rectangular to cylindrical, or rectangular to spherical, or rectangular to some other special case (from the region we are integrating over). The Jacobian represents a transformation of the variables, and it tells us how to rescale our integrals in order to obtain the correct value for the area/volume in the new coordinate system.

$$J(u,v)=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

$$x=g(u,v),y=h(u,v)$$

Example for polar.

$$x=r\cos\theta\,,y=r\sin\theta$$

$$J(r,\theta)=\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\cos^2\theta+r\sin^2\theta=r(\cos^2\theta+\sin^2\theta)=r$$

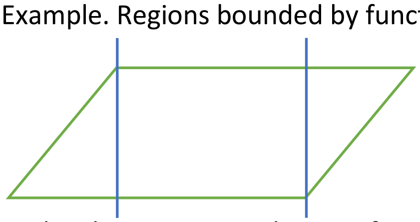
When I transform from rectangular coordinates to polar: $dA=|J(r,\theta)|drd\theta$

Example.
Find the Jacobian for the change of variables $x=uv,y=\frac{u}{v}$.

$$J(u,v)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}v & u\\ \frac{1}{v} & -\frac{u}{v^2}\end{vmatrix}=v\left(-\frac{u}{v^2}\right)-u\left(\frac{1}{v}\right)=-\frac{u}{v}-\frac{u}{v}=-\frac{2u}{v}$$

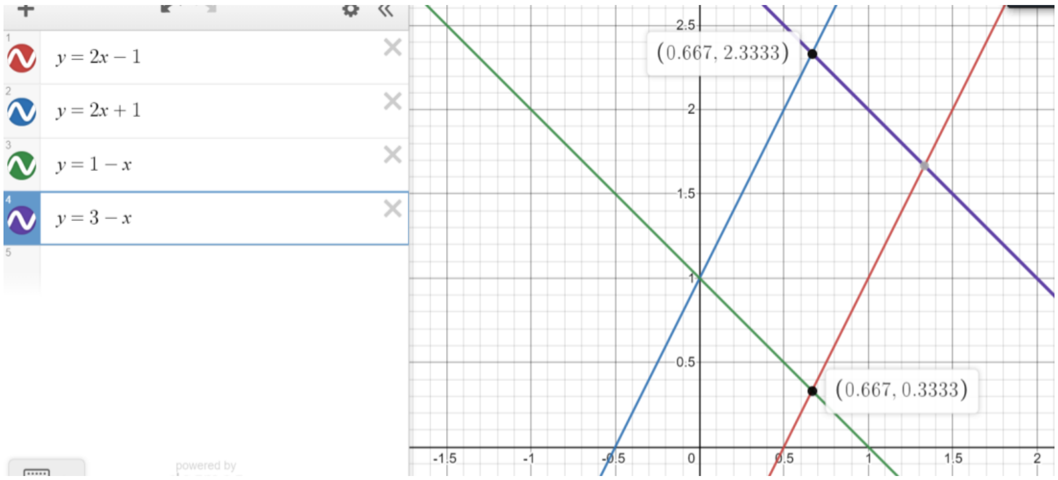
$$dA=\frac{2u}{v}dudv$$

Example. Regions bounded by functions, set up the coordinate transformation (find the Jacobian)



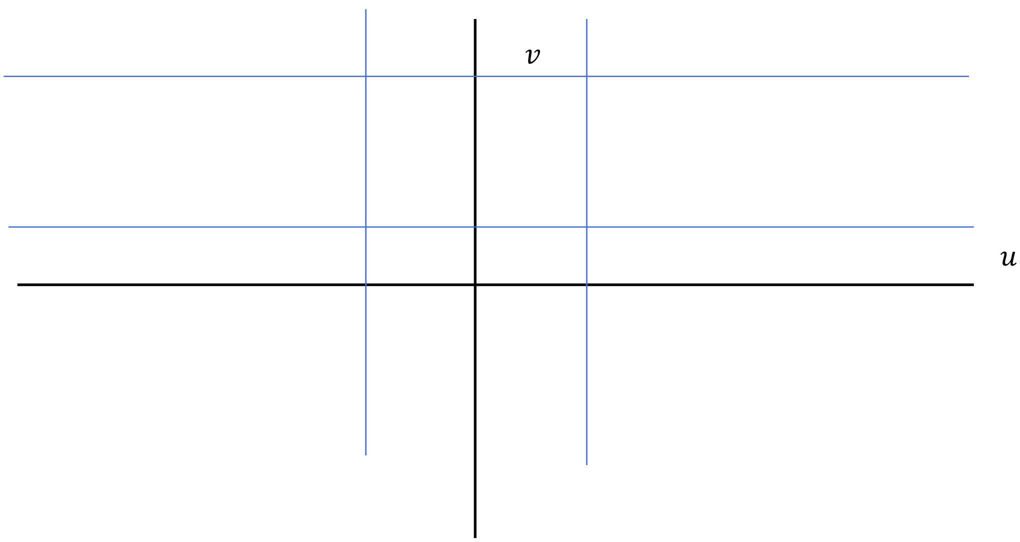
In rectangular coordinates, I would need 3 separate integrals owing to the changing top or bottom functions.

R is bounded by $y=2x-1,y=2x+1,y=1-x,y=3-x$



$$y = 2x - 1, y = 2x + 1, y = 1 - x, y = 3 - x$$

$1 = 2x - y$ $-1 = 2x - y$ $u = 2x - y$ $u \in [-1,1]$	$x + y = 1$ $x + y = 3$ $v = x + y$ $v \in [1,3]$



To find x and y in terms of u and v.

$$u = 2x - y$$
$$v = x + y$$

Add.

$$u + v = 3x$$
$$x = \frac{1}{3}(u + v)$$

$$u = 2x - y$$
$$-2v = -2x - 2y$$

Add.

$$u - 2v = -3y$$

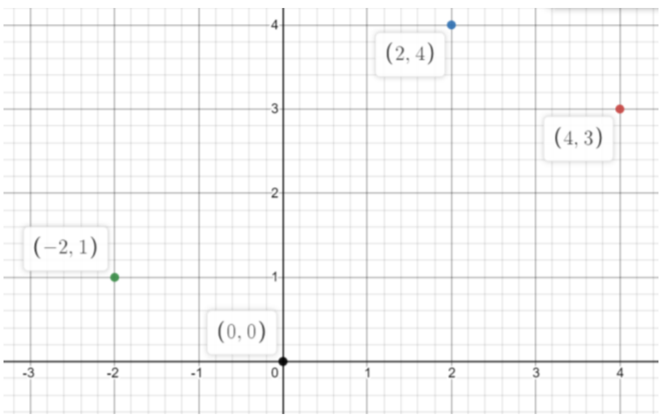
$$y = \frac{1}{3}(2v - u)$$

Find Jacobian.

$$J(u,v) = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{vmatrix} = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) - \left(-\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{9} + \frac{1}{9} = \frac{1}{3}$$

$$dA = \frac{1}{3} du dv$$

Example. From coordinates.
Region is inside a parallelogram with vertices (0,0), (4,3), (2,4) and (-2,1).



$$m_1 = \frac{4-1}{2-(-2)} = \frac{3}{4}$$

$$m_2 = \frac{3}{4}$$

$$m_3 = \frac{1}{-2} = -\frac{1}{2}$$

$$m_4 = \frac{4-3}{2-4} = \frac{1}{-2} = -\frac{1}{2}$$

$$y - 4 = \frac{3}{4}(x-2) \to 4y - 16 = 3x - 6 \to 3x - 4y = -10$$

$$y = \frac{3}{4}x \to 4y = 3x \to 3x - 4y = 0$$

$$y = -\frac{1}{2}x \to 2y = -x \to x + 2y = 0$$

$$y - 4 = -\frac{1}{2}(x-2) \to 2y - 8 = -x + 2 \to x + 2y = 10$$

$$u = 3x - 4y, u \in [-10,0]$$

$$v = x + 2y, v \in [0,10]$$

To find the Jacobian you would have to solve for x and y in terms of u and v.

Example.

Region is bounded by the hyperbolas $y = \frac{1}{x}, y = \frac{4}{x}, y = x, y = 4x$

$$xy = 1, xy = 4 \to u = xy, u \in [1,4]$$
$$\frac{y}{x} = 1, \frac{y}{x} = 4 \to v = \frac{y}{x}, v[1,4]$$

$$u = xy, v = \frac{y}{x}$$

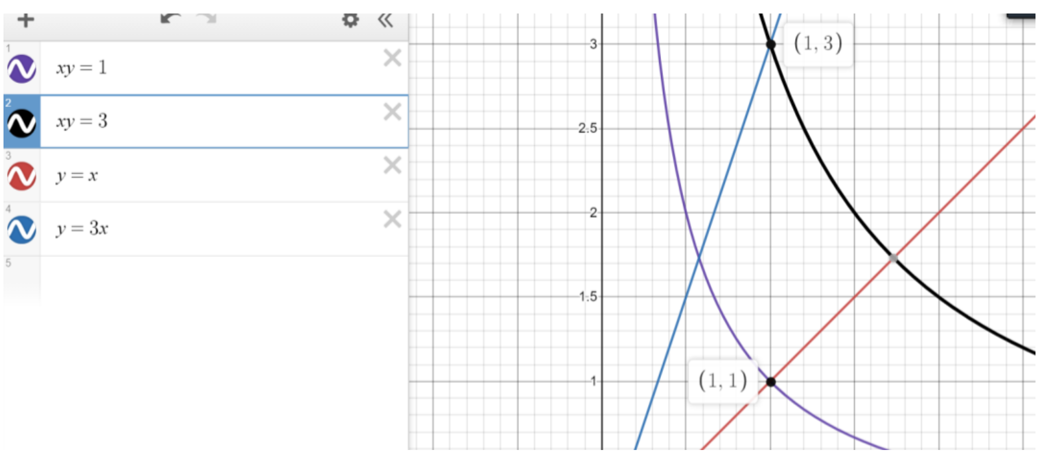
$$vx = y$$
$$u = x(vx) = vx^2$$
$$u = vx^2$$
$$x^2 = \frac{u}{v}$$
$$x = \sqrt{\frac{u}{v}} = u^{\frac{1}{2}}v^{-\frac{1}{2}}$$

$$u = xy$$
$$u = \sqrt{\frac{u}{v}}y$$
$$y = u\sqrt{\frac{v}{u}} = \sqrt{uv} = u^{\frac{1}{2}}v^{\frac{1}{2}}$$

Example.

$$\iint_R xy\, dA$$

Over the region in the first quadrant bounded by $y = x, y = 3x, xy = 1, xy = 3$



Suggested transformations:

$$x = \frac{u}{v}$$
$$y = v$$

$$\iint_R u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \iint_R \frac{u}{v} dudv$$

$$J(u,v) = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \left(\frac{1}{v}\right)(1) - (0)\left(-\frac{u}{v^2}\right) = \frac{1}{v}$$

Find the bounds on the region.

$$y = x, y = 3x, xy = 1, xy = 3$$
$$x = \frac{u}{v}$$
$$y = v$$

$$u = (xy) = \left(\frac{u}{v}\right)(v) \ \rightarrow u \in [1,3]$$

$$y = v \rightarrow x \le v \le 3x \rightarrow \frac{u}{v} \le v \le \frac{3u}{v} \rightarrow u \le v^2 \le 3u \rightarrow \sqrt{u} \le v \le \sqrt{3u}$$

$$\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}=\sqrt{3}\sqrt{u}} \frac{u}{v} dvdu$$

$$\int_1^3 u \ln v|_{\sqrt{u}}^{\sqrt{3u}} du = \int_1^3 u \ln\sqrt{3u} - u\ln\sqrt{u}\, du = \int_1^3 \frac{1}{2}u\ln 3u - \frac{1}{2}u\ln u\, du$$

To finish, you need integration by parts.

Three-variable case.

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$
$$= \frac{\partial x}{\partial u}\left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\frac{\partial z}{\partial v}\right) - \frac{\partial x}{\partial v}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\frac{\partial z}{\partial u}\right) + \frac{\partial x}{\partial w}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)$$

We want to calculate the Jacobian for spherical coordinates.

$$x = \rho\cos\theta\sin\phi, y = \rho\sin\theta\sin\phi, z = \rho\cos\phi$$

$$J(\rho,\theta,\phi) = \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} =$$

$\cos\theta\sin\phi\,(\rho\cos\theta\sin\phi\,(-\rho\sin\phi) - 0) - (-\rho\sin\theta\sin\phi)(\sin\theta\sin\phi\,(-\rho\sin\phi) - \cos\phi\,\rho\sin\theta\cos\phi) + \rho cos\theta\cos\phi\,(0 - \cos\phi\,\rho\cos\theta\sin\phi)$=

$$-\rho^2\cos^2\theta\sin^2\phi\, sin\phi - \rho^2\sin^2\phi\, sin\phi\sin^2\theta - \rho^2\cos^2\phi\sin\phi\sin^2\theta - \rho^2\sin\phi\cos^2\theta\cos^2\phi$$
$$-\rho^2\cos^2\theta\sin\phi\,(\sin^2\phi + \cos^2\phi) - \rho^2\sin^2\theta\sin\phi\,(\sin^2\phi + \cos^2\phi) =$$
$$-\rho^2\cos^2\theta\sin\phi - \rho^2\sin^2\theta\sin\phi =$$
$$-\rho^2\sin\phi\,(\cos^2\theta + \sin^2\theta) = -\rho^2\sin\phi$$

$$J(\rho,\theta,\phi) = |-\rho^2\sin\phi| = \rho^2\sin\phi$$

Centers of Mass.

Suppose you have a mass density function $\rho(x, y)$.
Total Mass: $M = \iint_R \rho(x, y)\, dA$

To find the center of mass is to multiply the density by the variable (direction) you want to find the center of mass in. The x-coordinate for the center of mass, multiple the density function by x. If you want to find the y-coordinate for the center of mass, multiply the density function by y. then re-integrate.

$$M_y = \iint_R x\rho(x, y)\, dA$$
$$M_x = \iint_R y\rho(x, y)\, dA$$

Center of Mass:

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

If you switch coordinate systems, algebraically replace x or y with your change of coordinates along with the density function. The limits of integration from the total mass calculation do not change. The results are still in rectangular coordinates.

In 3D:
Mass density function is $\rho(x, y, z)$
Total mass:

$$M = \iiint_Q \rho(x, y, z)\, dV$$

$$M_{xy} = \iiint_Q z\rho(x, y, z) dV$$
$$M_{xz} = \iiint_Q y\rho(x, y, z) dV$$

$$M_{yz} = \iiint_Q x\rho(x, y, z)dV$$

Center of mass:

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$$

In Probability:
We have probability density functions, instead of mass density functions.
To calculate total probabilities of continuous distributions, you integrate the density function to get the total probability. To find the mean, multiply the density function by x (or whatever variable you have) to obtain the mean in that direction.

For Thursday:
We'll do some examples of the center of mass problems.

Will review the final.
Also, if you are going to take the final early, remind me again this week.