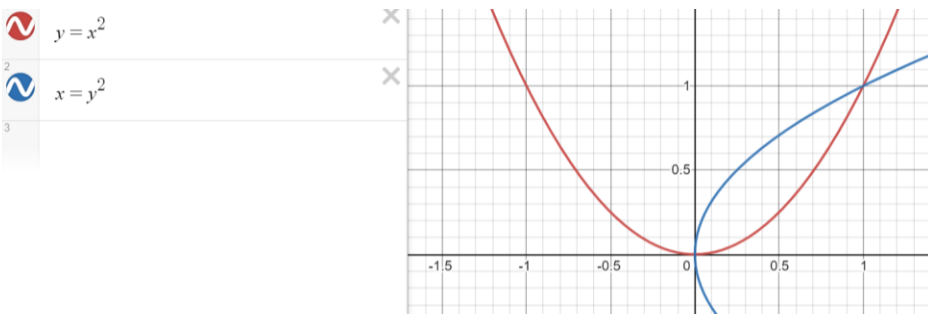
6/24/2021

Centers of Mass, Probability (15.5, 15.7-15.9)
Review for Final

Example.
The region is bounded by the parabolas $y = x^2, x = y^2$, and density is $\rho(x, y) = \sqrt{x}$.



$$M = \int_0^1 \int_{x^2}^{\sqrt{x}} \sqrt{x} dy\, dx = \int_0^1 \sqrt{x}(\sqrt{x} - x^2) dx = \int_0^1 x - x^{\frac{5}{2}} dx = \frac{1}{2}x^2 - \frac{2}{7}x^{\frac{7}{2}}\Big|_0^1 = \frac{1}{2} - \frac{2}{7} = \frac{3}{14}$$

$$M_y = \int_0^1 \int_{x^2}^{\sqrt{x}} x\sqrt{x} dy\, dx = \int_0^1 \int_{x^2}^{\sqrt{x}} x^{\frac{3}{2}} dy dx = \int_0^1 x^{\frac{3}{2}}(\sqrt{x} - x^2) dx = \int_0^1 x^2 - x^{\frac{7}{2}} dx = \frac{1}{3}x^3 - \frac{2}{9}x^{\frac{9}{2}}\Big|_0^1$$

$$= \frac{1}{3} - \frac{2}{9} = \frac{1}{9}$$

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{1}{9}}{\frac{3}{14}} = \frac{1}{9}\left(\frac{14}{3}\right) = \frac{14}{27}$$

$$M_x = \int_0^1 \int_{x^2}^{\sqrt{x}} y\sqrt{x} dy dx = \int_0^1 \frac{1}{2} y^2 \sqrt{x}\Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 \frac{1}{2}\sqrt{x}(x - x^4) dx = \int_0^1 \frac{1}{2} x^{\frac{3}{2}} - x^{\frac{9}{2}} dx =$$

$$\frac{1}{2}\left[\frac{2}{5}x^{\frac{5}{2}} - \frac{2}{11}x^{\frac{11}{2}}\right]_0^1 = \frac{1}{2}\left[\frac{2}{5} - \frac{2}{11}\right] = \frac{6}{55}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{6}{55}}{\frac{3}{14}} = \frac{6}{55}\left(\frac{14}{3}\right) = \frac{28}{55}$$

Center of Mass

$$\left(\frac{14}{27}, \frac{28}{55}\right)$$

Example.

Find the mass and center of mass of a solid hemisphere of radius 3, if the density at any point is proportional to is distance from the base.

$$\rho(x, y, z) = kz$$

Work in spherical.

$$M = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^3 \rho \cos\phi \, \rho^2 \sin\phi d\rho d\phi d\theta$$

$$M_{xy} = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^3 \rho \cos\phi \, \rho \cos\phi \, \rho^2 \sin\phi d\rho d\phi d\theta \, , \bar{z} = \frac{M_{xy}}{M}$$

$$M_{xz} = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^3 \rho \sin\theta \sin\phi \, \rho \cos\phi \, \rho^2 \sin\phi d\rho d\phi d\theta \, , \bar{y} = \frac{M_{xz}}{M}$$

$$M_{yz} = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^3 \rho \cos\theta \, sin\phi \, \rho \cos\phi \, \rho^2 \sin\phi d\rho d\phi d\theta \, , \bar{x} = \frac{M_{yz}}{M}$$

Center of Mass/centroid $(\bar{x}, \bar{y}, \bar{z})$

Probability questions on the homework.
If you have a probability density function, the area under the curve for all possible values of x, or (x,y), or (x,y,z), must total 1.

$$f(x) = kx^2, 0 \leq x \leq 3$$

Find k to make the integral equal to 1.

$$\int_0^3 kx^2 \, dx = \frac{k}{3}x^3\Big|_0^3 = \frac{k}{3}(27) = 9k$$

$$9k = 1 \rightarrow k = \frac{1}{9}$$

What is the probability of being in some range of values, like $[0,1]$? $P(0 \leq X \leq 1)$.

$$\int_0^1 \frac{1}{9}x^2 dx = \frac{1}{27}x^3\Big|_0^1 = \frac{1}{27}$$

All probability values should be between 0 and 1.

Find the expected value/mean: multiply the probability density function by x (and divide by the total probability).

$$\bar{x} = \int_0^3 x\left(\frac{1}{9}x^2\right) dx = \int_0^3 \frac{1}{9}x^3 dx = \frac{1}{36}x^4\Big|_0^3 = \frac{81}{36} = \frac{9}{4}$$

Probability with two variables:

$$f(x, y) = 4xy, 0 \leq x \leq 1, 0 \leq y \leq 1$$

Outside the given range, the probability is zero.

$$\int_0^1 \int_0^1 4xydydx = 1$$

$$P\left(X \geq \frac{1}{2}\right) = \int_{\frac{1}{2}}^1 \int_0^1 4xydydx$$

$$P\left(X \geq \frac{1}{2}, Y \leq \frac{1}{2}\right) = \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} 4xydydx$$

Expected value for x is

$$\bar{x} = \int_0^1 \int_0^1 (x)4xydydx$$

The expected value for y is

$$\bar{y} = \int_0^1 \int_0^1 (y)4xydydx$$

Final exam is Monday!