6/8/2021

Tangents

Tangents and Normal vectors of space curves (13.2,13.3)

Tangent Planes, Normal Vectors to surfaces, Directional Derivatives (14.6)

Space curves in 2D and in 3D

Define the tangent to a curve in space.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

The definition of the unit tangent vector.

The tangent line can be constructed without making the vector a unit vector, but we will need the unit vector to construct the normal vector (unit), and the binormal vector. These three vectors will form a basis for the 3D space that moves along the curve with the particle.

$$r(t) = \langle 1 + t^3, te^{-t}, \sin 2t \rangle$$

$$r'(t) = \langle 3t^2, e^{-t} - te^{-t}, 2\cos 2t \rangle$$

$$T(t) = \frac{\langle 3t^2, e^{-t} - te^{-t}, 2\cos 2t \rangle}{\sqrt{9t^4 + (e^{-t} - te^{-t})^2 + 4\cos^2 2t}} = \frac{\langle 3t^2, e^{-t} - te^{-t}, 2\cos 2t \rangle}{\sqrt{9t^4 + e^{-2t}(1-t)^2 + 4\cos^2 2t}}$$

$$= \frac{\langle 3t^2, e^{-t} - te^{-t}, 2\cos 2t \rangle}{\sqrt{9t^4 + e^{-2t}(1 - 2t + t^2) + 4\cos^2 2t}}$$

Most space curves look awful as a unit tangent vector.

Example of a helix.

$$r(t) = \langle 2\cos 3t\,, 2\sin 3t\,, 5t \rangle$$

$$r'(t) = \langle -6\sin 3t\,, 6\cos 3t\,, 5 \rangle$$

$$T(t) = \frac{\langle -6\sin 3t\,, 6\cos 3t\,, 5 \rangle}{\sqrt{36\sin^2 3t + 36\cos^2 3t + 25}} = \frac{\langle -6\sin 3t\,, 6\cos 3t\,, 5 \rangle}{\sqrt{36(\sin^2 3t + \cos^2 3t) + 25}}$$

$$= \frac{\langle -6\sin 3t\,, 6\cos 3t\,, 5 \rangle}{\sqrt{36 + 25}} = \frac{\langle -6\sin 3t\,, 6\cos 3t\,, 5 \rangle}{\sqrt{61}} = \langle -\frac{6}{\sqrt{61}}\sin 3t\,, \frac{6}{\sqrt{61}}\cos 3t\,, \frac{5}{\sqrt{61}} \rangle$$

What is the equation of the tangent line on the helix at $t = \frac{\pi}{4}$

Find the point on the curve: $\langle -\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, \frac{5\pi}{4} \rangle$.

Find the tangent vector: $\langle -\frac{12}{\sqrt{122}}, -\frac{12}{\sqrt{122}}, \frac{5}{\sqrt{61}} \rangle$

The equation of the tangent line $g(t) = \langle -\frac{12}{\sqrt{122}}t - \frac{2}{\sqrt{2}}, -\frac{12}{\sqrt{122}}t + \frac{2}{\sqrt{2}}, \frac{5}{\sqrt{61}}t + \frac{5\pi}{4}\rangle$

Unit normal vector is the second component of our traveling coordinate system
Normal points in a direction perpendicular to the tangent vector. The normal vector either points toward the center of the curvature, or it away from the center of curvature. The principal normal vector (here) is the normal vector pointing toward the inside of the curve (in a circle, this would point toward the center)

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

$$T(t) = \langle -\frac{6}{\sqrt{61}}\sin 3t, \frac{6}{\sqrt{61}}\cos 3t, \frac{5}{\sqrt{61}}\rangle$$

$$T'(t) = \langle -\frac{18}{\sqrt{61}}\cos 3t, -\frac{18}{\sqrt{61}}\sin 3t, 0\rangle$$

$$N(t) = \frac{\langle -\frac{18}{\sqrt{61}}\cos 3t, -\frac{18}{\sqrt{61}}\sin 3t, 0\rangle}{\sqrt{\frac{324}{61}\cos^2 3t + \frac{324}{61}\sin^2 3t}} = \frac{\langle -\frac{18}{\sqrt{61}}\cos 3t, -\frac{18}{\sqrt{61}}\sin 3t, 0\rangle}{\sqrt{\frac{324}{61}}}$$

$$= \frac{\langle -\frac{18}{\sqrt{61}}\cos 3t, -\frac{18}{\sqrt{61}}\sin 3t, 0\rangle}{\frac{18}{\sqrt{61}}} = \frac{\sqrt{61}}{18}\langle -\frac{18}{\sqrt{61}}\cos 3t, -\frac{18}{\sqrt{61}}\sin 3t, 0\rangle = \langle -\cos 3t, -\sin 3t, 0\rangle$$

This is a unit vector.

The third component is called the binormal vector because it is perpendicular to both the tangent vector and the normal vector.

$$B(t) = T(t) \times N(t)$$

$$\begin{vmatrix} i & j & k \\ -\frac{6}{\sqrt{61}}\sin 3t & \frac{6}{\sqrt{61}}\cos 3t & \frac{5}{\sqrt{61}} \\ -\cos 3t & -\sin 3t & 0 \end{vmatrix} = \left(\frac{5}{\sqrt{61}}\sin 3t\right)i - \left(\frac{5}{\sqrt{61}}\cos 3t\right)j + \left(\frac{6}{\sqrt{61}}\sin^2 3t + \frac{6}{\sqrt{61}}\cos^2 3t\right)k$$

$$= \langle \frac{5}{\sqrt{61}}\sin 3t, -\frac{5}{\sqrt{61}}\cos 3t, \frac{6}{\sqrt{61}}\rangle$$

Binormal is always already a unit vector if you use the unit tangent vector and the unit normal vector to find it.

For 2D curves: if your unit tangent vector is of the form $\langle x(t), y(t)\rangle$, then the unit normal vector will either have the form $\langle -y(t), x(t)\rangle$ or $\langle y(t), -x(t)\rangle$. What you don't know in advance is which of these is the principal normal vector.

Tangents for surfaces.

Start with directional derivatives.
Our partial derivatives for a surface $z = f(x, y)$, told us the way that x was changing when y was held constant, or how y was changing if x was held constant.

Find the gradient of the surface: $\nabla f = \langle f_x, f_y\rangle$, and we going to need the direction in which we are heading, $u = \langle \cos\theta, \sin\theta\rangle$ or any other unit vector.

$$D_u = \nabla f \cdot u$$

Find the directional derivative for the function $f(x, y) = e^x \sin y$ at the point $\left(0, \frac{\pi}{3}\right)$ in the direction of $v = \langle -6,8\rangle$.

$$\nabla f = \langle e^x \sin y, e^x \cos y\rangle$$
$$\nabla f\left(0, \frac{\pi}{3}\right) = \langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle$$

Find u for the direction. $\|v\| = \sqrt{36 + 64} = \sqrt{100} = 10$, so $u = \langle -\frac{6}{10}, \frac{8}{10}\rangle = \langle -\frac{3}{5}, \frac{4}{5}\rangle$

$$D_u = \langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle \cdot \langle -\frac{3}{5}, \frac{4}{5}\rangle = \frac{\sqrt{3}}{2}\left(-\frac{3}{5}\right) + \frac{1}{2}\left(\frac{4}{5}\right) = -\frac{3\sqrt{3}}{10} + \frac{4}{10} = \frac{4 - 3\sqrt{3}}{10} \approx -0.1196$$

Example in 3D.

$$\nabla f = \langle f_x, f_y, f_z\rangle$$

$f(x, y, z) = xe^y + ye^z + ze^x$, at the point (0,0,0) and in the direction $v = \langle 5,1,-2\rangle$.

$$\nabla f = \langle e^y + ze^x, xe^y + e^z, ye^z + e^x\rangle$$
$$\nabla f(0,0,0) = \langle 1,1,1\rangle$$

$$\|v\| = \sqrt{25 + 1 + 4} = \sqrt{30}$$
$$u = \langle \frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}, -\frac{2}{\sqrt{30}}\rangle$$

$$D_u = \langle 1,1,1\rangle \cdot \langle \frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}, -\frac{2}{\sqrt{30}}\rangle = \frac{5}{\sqrt{30}} + \frac{1}{\sqrt{30}} - \frac{2}{\sqrt{30}} = \frac{4}{\sqrt{30}}$$

The gradient always points in the direction of maximum increase. Amount of that maximum increase is the magnitude of the gradient at the point, $\|\nabla f\|$.

And the negative of the gradient always points in the direction of the maximum decrease, $-\nabla f$.

Tangent planes.
Curves have tangent lines, but surfaces have tangent surfaces (planes).

Tangent planes use a slightly different gradient than does the directional derivative. To orient our tangent plane in three-dimensions, we'll need a three-dimensional vector, even though the surface is a function of only x and y. $f(x, y) = z$.

Construct a three-variable function from our two-variable function.

$$z = f(x, y)$$

Move everything to one side of the equation, instead of 0, set the result equal to $F(x, y, z)$.

$f(x, y) = 3y^2 - 2x^2 + x$, at the point $(2, -1, -3)$

$$z = 3y^2 - 2x^2 + x$$
$$F(x, y, z) = 3y^2 - 2x^2 + x - z$$

The vector ∇F is a vector which is perpendicular (normal) to the surface at the point where it is evaluated.

$$\nabla F = \langle -4x + 1, 6y, -1 \rangle$$

$$\nabla F(2, -1, -3) = \langle -7, -6, -1 \rangle$$

$$-7(x - 2) - 6(y + 1) - 1(z + 3) = 0$$

Equation of the tangent plane. Could solve for z if we wanted.

Example.

$$x^2 + y^2 + z^2 = 9$$

$$F(x, y, z) = x^2 + y^2 + z^2 - 9$$

$$\nabla F = \langle 2x, 2y, 2z \rangle$$

$$@(1,2,2)$$

Normal vector for plane $\nabla F(1,22) = \langle 2,4,4 \rangle$

Tangent plane equation:

$$2(x - 1) + 4(y - 2) + 4(z - 2) = 0$$

If the surface is expressed in cylindrical or spherical coordinates: convert back to rectangular to find the tangent plane (or use the appropriate gradient formula in the Del Notation handout).

What about parametric surface form? (16.6)

How do we get a vector perpendicular to a parametric surface?

Key: the partial derivatives for a parametric surface are vectors in the plane.

$r(u, v)$ is the surface, r_u and r_v are vectors in plane. So we need a vector perpendicular to both r_u and r_v. So we find that by doing the cross product of the two vectors.

Example.

$$r(u, v) = \langle u^2, 2u \sin v, u \cos v \rangle, u = 1, v = 0$$

(If you are given a point in 3D space instead of values of u and v, set your vector components equal to the point components, and solve the set of simultaneous equations.)

Point on the surface $r(1,0) = \langle 1,0,1 \rangle$

$$r_u = \langle 2u, 2 \sin v, \cos v \rangle$$
$$r_v = \langle 0,2u \cos v, -u \sin v \rangle$$

$$N(u, v) = r_u \times r_v = \begin{vmatrix} i & j & k \\ 2u & 2 \sin v & \cos v \\ 0 & 2u \cos v & -u \sin v \end{vmatrix} =$$

$$(-2u \sin^2 v - 2u \cos^2 v)i - (-2u^2 \sin v - 0)j + (4u^2 \cos v - 0)k =$$

$$\langle -2u, 2u^2 \sin v, 4u^2 \cos v \rangle$$

Normal vector to the surface at the point (1,0,1)

$$N = \langle -2,0,4 \rangle$$

Tangent plane equation is

$$-2(x - 1) + 0(y - 0) + 4(z - 1) = 0$$

Tangents and Normals handout. It covers tangents for curves, tangent planes in rectangular coordinates and tangent planes from parameterized surfaces.

You can construct a "normal line" to the surface. When you find the normal vector to the surface, make a line of it using the normal vector as the direction, and the point on the plane as the point on the line (x_0, y_0, z_0).