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Arc Length and Curvature (13.3)
Surface Area (15.6,16.6)

A line integral without an inside function is the length of the curve only.

$$L = \int_a^b ds = \int_a^b |r'(t)|dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt$$

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

Example.

$$r(t) = \langle \cos t , \sin t , t \rangle$$

Find the length of one turn of the helix, from 0 to 2π

$$r'(t) = \langle -\sin t , \cos t , 1 \rangle$$

$$\|r'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{1+1} = \sqrt{2}$$

$$\int_0^{2\pi} ds = \int_0^{2\pi} \sqrt{2}dt = 2\sqrt{2}\pi$$

Example.

$$r(t) = \langle t, t^2, t^3 \rangle$$

$$r'(t) = \langle 1,2t, 3t^3 \rangle$$

$$\|r'(t)\| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\int_a^b \sqrt{1 + 4t^2 + 9t^4}dt$$

Can only be evaluated numerically.

Example.

$$r(t) = \langle 1, t^2, t^3 \rangle$$

$$r'(t) = \langle 0,2t, 3t^3 \rangle$$

$$\|r'(t)\| = \sqrt{4t^2 + 9t^4} = \sqrt{t^2(4 + 9t^2)} = t\sqrt{4 + 9t^2}$$

$$\int_a^b t\sqrt{4 + 9t^2}dt$$

This one can be done by hand using u-substitution

Curvature
Estimating how quickly or slowly the graph is turning/curving by approximating the curve with a circle.

Definition: $\kappa = \left|\frac{dT}{ds}\right|$. This is the derivative of the unit tangent vector taken with respect to the arc length.

Converting a vector-valued function from being expressed in terms of time to being expressed in terms of arclength is usually impossible.

To do the conversion we would need to have a function that relates the arclength to time, and then solve for time in terms of the arclength.

$$s(t) = \int_a^t ds = \int_a^t \|r'(u)\| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

$s(t) = \int_0^t \sqrt{2} du = \sqrt{2}t \rightarrow t = \frac{s}{\sqrt{2}}$ (helix)

$$s(t) = \int_a^t \sqrt{1 + 4u^2 + 9u^4} du$$

$$s(t) = \int_a^t u\sqrt{4 + 9u^2} du$$

Solving for t with respect to s is not always doable, but it is for a helix.

$$r(t) = \langle \cos t, \sin t, t \rangle$$

$$r(s) = \langle \cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \rangle$$

Each unit of "time" is now one unit of arclength.
Easy to apply definition of curvature from here.

$$T(s) = r'(s)$$

$$T(s) = \langle -\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$$

$$\kappa = |T'(s)| = \left|\langle -\frac{1}{2}\cos\frac{s}{\sqrt{2}}, -\frac{1}{2}\sin\frac{s}{\sqrt{2}}, 0 \rangle\right| = \sqrt{\frac{1}{4}\cos^2\frac{s}{\sqrt{2}} + \frac{1}{4}\sin^2\frac{s}{\sqrt{2}}} = \frac{1}{2}$$

Radius of curvature is the radius of a circle that can be used to estimate the curvature of the graph: $R \approx \frac{1}{\kappa}$. For this helix, the radius of curvature is 2.

Most of the time the formal definition of curvature is not helpful, so we need a definition that depends on functions of t not s.

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

Use this formula for all space curves or parameterized curves, unless they are already defined in terms of the arclength.

If we have a 2D function, $y = f(x)$, there is a version of this formula derived from parametrizing the curve.

$$\kappa(x) = \frac{|f''(x)|}{\left[1 + \left(f'(x)\right)^2\right]^{\frac{3}{2}}}$$

Example.

$$r(t) = \langle t, t^2, e^t \rangle$$
$$r'(t) = \langle 1, 2t, e^t \rangle$$
$$r''(t) = \langle 0, 2, e^t \rangle$$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 1 & 2t & e^t \\ 0 & 2 & e^t \end{vmatrix} = (2te^t - 2e^t)i - (e^t - 0)j + (2-0)k = \langle 2e^t(t-1), -e^t, 2 \rangle$$

$$\|r'(t) \times r''(t)\| = \sqrt{4e^{2t}(t^2 - 2t + 1) + e^{2t} + 4}$$

$$\|r'(t)\|^3 = \left(\sqrt{2 + 4t + e^{2t}}\right)^3$$

$$\kappa = \frac{\sqrt{4e^{2t}(t^2 - 2t + 1) + e^{2t} + 4}}{\left(\sqrt{2 + 4t + e^{2t}}\right)^3}$$

What is the curvature at $t = 0$? What is the radius of curvature at the same point?

$$\kappa(0) = \frac{\sqrt{4e^0(0^2 - 2(0) + 1) + e^0 + 4}}{\left(\sqrt{2 + 4(0) + e^0}\right)^3} = \frac{\sqrt{9}}{\sqrt{3}^3} = \frac{1}{\sqrt{3}}$$

$$R \approx \frac{1}{\kappa} = \sqrt{3}$$

Example.

Find the curvature of the function $y = x^4$.

$$f'(x) = 4x^3, f''(x) = 12x^2$$

$$\kappa = \frac{12x^2}{\left(\sqrt{1 + (4x^3)^2}\right)^3} = \frac{12x^2}{(1 + 16x^6)^{\frac{3}{2}}}$$

$$\kappa(1) = \frac{12}{(17)^{\frac{3}{2}}}$$

Surface Area (15.6)

In the textbook, the formula for surface area is given as (for functions $z = f(x, y)$):

$$A(S) = \iint_R dS = \iint_R \sqrt{(f_x)^2 + \left(f_y\right)^2 + 1dA}$$

Recall that $s = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx$

When we found the normal vector to the tangent plane $\nabla F = \langle f_x, f_y, -1 \rangle \text{ or } \langle -f_x, -f_y, 1 \rangle$

What is the magnitude of this vector? $\|\nabla F\| = \sqrt{f_x^2 + f_y^2 + 1}$

$$A(S) = \iint_R \|\nabla F\| dA = \iint_R \|N\| dA$$

Example.

Find the surface area of the paraboloid $x^2 + y^2 = z$, that lies under the plane $z = 9$.

The region we are integrating over $x^2 + y^2 = 9$. A circle of radius 3.

$$F(x, y, z) = x^2 + y^2 - z$$
$$\nabla F = \langle 2x, 2y, -1 \rangle$$
$$\|\nabla F\| = \sqrt{4x^2 + 4y^2 + 1}$$

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{4x^2 + 4y^2 + 1}\, dydx$$

In polar:

$$\int_0^{2\pi} \int_0^3 \sqrt{4r^2 + 1} r dr d\theta$$

$$u = 4r^2 + 1, du = 8rdr, \frac{1}{8} du = rdr$$

$$\int_0^{2\pi} \int_1^{37} \frac{1}{8} u^{\frac{1}{2}} du d\theta = \int_0^{2\pi} \frac{1}{8} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_1^{37} d\theta = \int_0^{2\pi} \frac{1}{12} \left(37^{\frac{3}{2}} - 1\right) d\theta = \frac{2\pi}{12} \left(37^{\frac{3}{2}} - 1\right) = \frac{\pi}{6} \left(37^{\frac{3}{2}} - 1\right)$$

The integral is the magnitude of the normal vector (the one you use for tangent planes, not directional derivative), and you will often have to switch to polar to integrate.

The one caveat is that this formula is for **functions**. When we were finding the tangent plane for a sphere, we used the implicit version of the formula to find the tangent plane, and F. We can't do that here. You will need to solve for z, and you can use symmetry to find the other half.