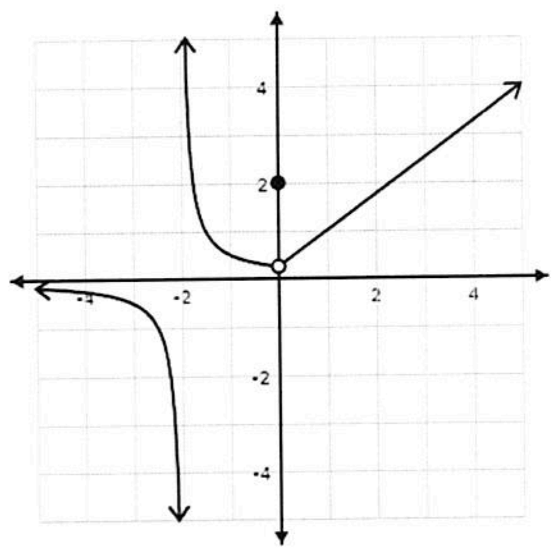
MTH 263, Exam #1, Summer 2023 Name KEY

Instructions: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error or any kind.

1. Find the indicated limits.
   a. $\lim_{x\to 4} \frac{x^3-64}{x-4}$ (8 points) $= \lim_{x\to 4} \frac{\cancel{(x-4)}(x^2+4x+16)}{\cancel{x-4}} = \lim_{x\to 4} x^2+4x+16 = 48$

   b. Consider the graph of the function of $f(x)$ below and then answer questions about limits of $f(x) = \begin{cases} \frac{1}{2(x+2)}, & x<0 \\ 2, & x=0 \\ \frac{1}{4}(3x+1), & x>0 \end{cases}$. (2 points each)



   i. $\lim_{x\to 0^-} f(x)$ $1/4$

   ii. $\lim_{x\to 0^+} f(x)$ $1/4$

   iii. $\lim_{x\to 0} f(x)$ $1/4$

   iv. $f(0)$ 2

   v. $\lim_{\cancel{x\to 2^-}} f(x)$ $-\infty$
   $x\to -2^-$

   vi. $\lim_{\cancel{x\to 2^+}} f(x)$ ∞
   $x\to -2^+$

   vii. $\lim_{\cancel{x\to 2}} f(x)$ DNE
   $x\to -2$

2. Consider the piecewise function $f(x) = \begin{cases} x^2 - 3x, x \le -1 \\ x + b, \quad x > -1 \end{cases}$. Determine for what value(s) of b is the function continuous? (5 points)

$$x^2 - 3x$$
$$(1)^2 - 3(1) = 1 - 3 = -2$$
$$1 + b = -2$$
$$b = -3$$

3. A particle is moving along a trajectory defined by $s(t) = 2t^2 - 4t + 10$. Find the average velocity on the interval $[0,2]$. (10 points)

$$s(0) = 10$$
$$s(2) = 2(2)^2 - 4(2) + 10 = 8 - 8 + 10 = 10$$

$$\frac{10-10}{2-0} = 0$$

4. Find the value of the indicated derivative. (8 points each)

a. $f(x) = 2x^2 - \frac{4}{x^3}, f'(1)$

$$2x^2 - 4x^{-3}$$

$$f'(x) = 4x + 16x^{-4} = 4x + \frac{16}{x^4}$$

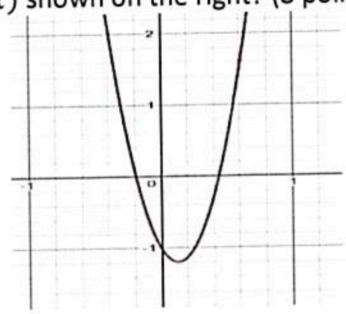
$$f'(1) = 4(1) + \frac{16}{1^4} = 4 + 16 = 20$$

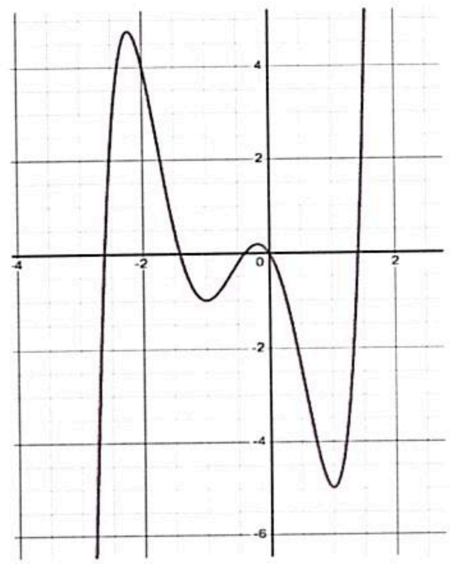
b. $f(x) = (2 - x^2)^4, f''(-1)$

$$f'(x) = 4(2-x^2)^3(-2x) = -8x(2-x^2)^3$$
$$f''(x) = -8(2-x^2)^3 - 8x(2-x^2)^2(-2x)$$
$$-8(2-x^2)^3 + 16x^2(2-x^2)^2$$

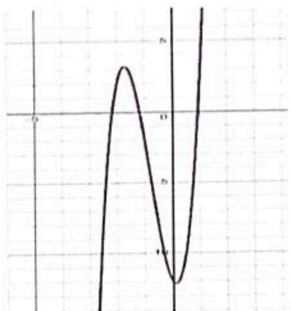
$$f''(-1) = -8(2-(-1)^2)^3 + 16(-1)^2(2-(-1)^2)^2 =$$
$$-8(1)^3 + 16(1)(1)^2 = -8 + 16 = 8$$

5. Which of the following graphs is most likely to be the graph of the derivative of the graph of $f(x)$ shown on the right? (8 points)

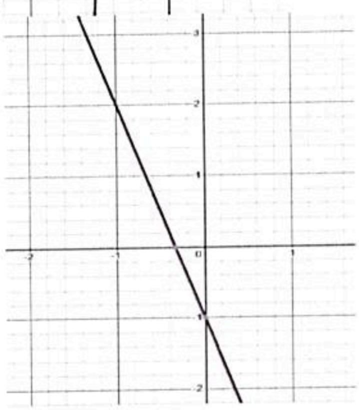




a.



b.



c.

d.

Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

6. Find the indicated limits. You may not use L'Hôpital's method for these problems. You may use algebraic methods or numerical methods.

a. $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}$ (12 points) $= \lim_{x\to 4}\frac{(\cancel{\sqrt{x}-2})(\sqrt{x}+2)}{\cancel{\sqrt{x}-2}} = \lim_{x\to 4}\sqrt{x}+2 = \sqrt{4}+2 = 2+2 = 4$

b. $\lim_{x\to 0} x\ln x$ (10 points) $= 0$

x	0.1	0.01	0.001	0.0001	0.00001
$f(x)$	-0.2303	-0.0461	-0.0069	-9×10^{-4}	1×10^{-4}

7. Use the squeeze theorem to prove that $\lim_{x \to 0} x^3 \cos\left(\frac{1}{x}\right) = 0$. (15 points)

$$-1 \le \cos\left(\tfrac{1}{x}\right) \le 1$$

$$-x^3 \le x^3 \cos\left(\tfrac{1}{x}\right) \le x^3$$

$$\lim_{x \to 0} -x^3 \le \lim_{x \to 0} x^3 \cos\left(\tfrac{1}{x}\right) \le \lim_{x \to 0} x^3$$

$$0 \le \lim_{x \to 0} x^3 \cos\left(\tfrac{1}{x}\right) \le 0$$

$$\therefore \lim_{x \to 0} x^3 \cos\left(\tfrac{1}{x}\right) = 0$$

8. A particle is moving along a trajectory defined by $s(t) = 8 + 9t - t^2$. Find the instantaneous velocity at the point $t = 1$ using the limit definition of the derivative. You should provide both the general equation, and the velocity at that point. (15 points)

$$S'(t) = 9 - 2t$$

$$S'(1) = 9 - 2 = 7$$

$$S'(t) = \lim_{h \to 0} \frac{8 + 9(t+h) - (t+h)^2 - (8 + 9t - t^2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{8} + \cancel{9t} + 9h - \cancel{t^2} - 2th - h^2 - \cancel{8} - \cancel{9t} + \cancel{t^2}}{h}$$

$$\lim_{h \to 0} \frac{9h - 2th - h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(9 - 2t - 2h)}{\cancel{h}} = \lim_{h \to 0} 9 - 2t - 2h =$$

$$9 - 2t$$

9. Use the $\varepsilon - \delta$ definition of the limit to prove that $\lim_{x \to 3} \left(\frac{1}{3}x + 7\right) = 8$. (15 points)

if $|x-3| < \delta$ and suppose that $\delta = 3\varepsilon$, then

$|x-3| < 3\varepsilon \rightarrow \frac{1}{3}|x-3| < \varepsilon \rightarrow \left|\frac{1}{3}x - 1\right| < \varepsilon \rightarrow$

$\left|\frac{1}{3}x + 7 - 8\right| < \varepsilon$. Therefore 8 is the $\lim_{x \to 3} \frac{1}{3}x + 7$.

$\left|\frac{1}{3}x + 7 - 8\right| = \left|\frac{1}{3}x - 1\right| =$

$\frac{1}{3}|x-3| < \varepsilon$

$|x3| < 3\varepsilon$

10. Find the derivative of each of the following. If no derivative notation is specified, find the first derivative only. If a derivative notation is specified with the function, find the indicated derivative. (10 points each)

a. $f(x) = e^{\tan x}$

$f'(x) = \sec^2 x \, e^{\tan x}$

b. $f(x) = x \tanh x, f''(x)$

$f'(x) = \tanh x + x \operatorname{sech}^2 x$

$f''(x) = \operatorname{sech}^2 x + \operatorname{sech}^2 x + x \cdot 2 \operatorname{sech} x \cdot \operatorname{sech}^2 x \tanh x \, (-1)$

$= 2 \operatorname{sech}^2 x - 2x \operatorname{sech}^2 x \tanh x$

c. $f(x) = \ln(arcsec\, x)$

$\frac{1}{\operatorname{arcsec} x} \cdot \frac{1}{x\sqrt{x^2-1}}$

d. $xy^2 + y = y \sin x, \frac{dy}{dx}$

$y^2 + 2xyy' + y' = y' \sin x + y \cos x$

$y'(2xy + 1 - \sin x) = y \cos x - y^2$

$\frac{dy}{dx} = \frac{y \cos x - y^2}{2xy + 1 - \sin x}$

e. $f(x) = (\tan x)^x$

$y = (\tan x)^x$

$\ln y = \ln (\tan x)^x$

$\ln y = x \ln \tan x$

$\frac{1}{y} y' = \ln \tan x + x \cdot \frac{1}{\tan x} \cdot \sec^2 x$

$y' = f'(x) = y \left[\ln \tan x + \frac{x \sec^2 x}{\tan x} \right]$

$f'(x) = (\tan x)^x \left[\ln \tan x + \frac{x \sec^2 x}{\tan x} \right]$

f. $f(x) = 2^x - \log_3(x^9 + \pi)$

$$f'(x) = (\ln 2)2^x - \frac{1}{\ln 3} \cdot \frac{1}{x^9+\pi} \cdot 9x^8$$

11. Find the equation of the tangent line to graph $y = \sqrt[4]{x^3+6}$ at $x = 1$. (15 points)

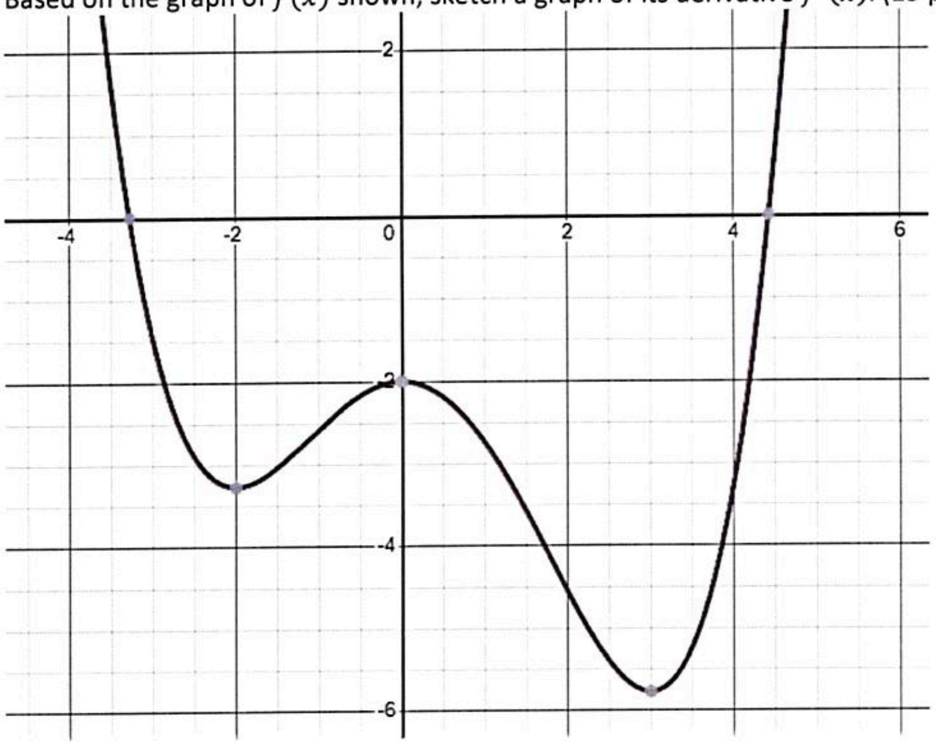
$$y' = (x^3+6)^{-3/4}(3x^2) = \frac{3x^2}{\sqrt[4]{(x^3+6)^3}}$$

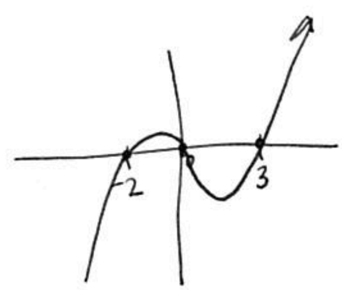
$$f'(1) = \frac{3(1)^2}{\sqrt[4]{(1+6)^3}} = \frac{3}{7^{3/4}}$$

$$f(1) = \sqrt[4]{7}$$

$$y - \sqrt[4]{7} = \frac{3}{7^{3/4}}(x-1)$$

12. Based on the graph of $f(x)$ shown, sketch a graph of its derivative $f'(x)$. (15 points)





13. Determine whether the Mean Value Theorem applies to the function $f(x) = x^3 + 2x + 1$ on the interval $[0,6]$. If it does not apply, explain why not. If it does apply, find the point in the interval where the slope of the tangent line is the same as the average rate of change over the entire interval. (6 points)

continuous   it applies

$f(0) = 0 + 0 + 1 = 1$

$f(6) = 216 + 12 + 1 = 229$

$\frac{229-1}{6-0} = \frac{228}{6} = 38$

$f'(x) = 3x^2 + 2$

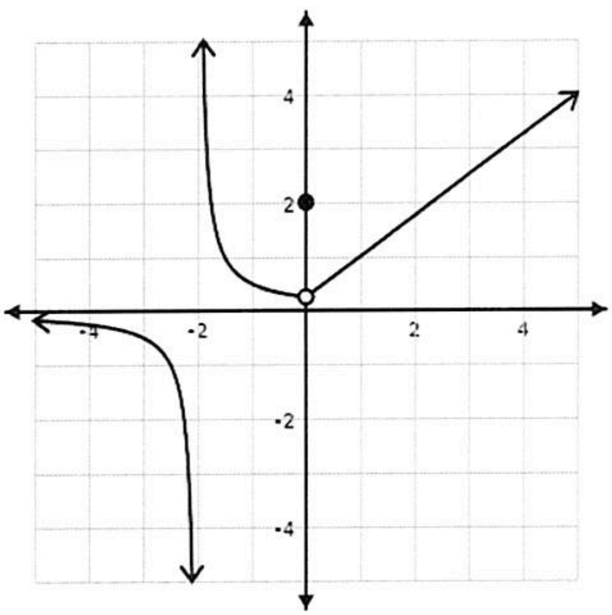
$38 = 3x^2 + 2$

$36 = 3x^2$

$12 = x^2$

$x = \sqrt{12} = 2\sqrt{3}$ (neg. not on interval)

14. For the function shown in the graph below, identify the location of each discontinuity, and classify it as a) a jump discontinuity, b) an infinite discontinuity, c) a removable discontinuity. Since only two of these can be represented in this graph (since there are only two discontinuities), draw a sketch of a function that contains the missing type. (10 points)



discontinuity at $x = -2$

infinite

discontinuity at $x = 0$

removeable

a jump discontinuity is like