MTH 263, Homework #4, Summer 2023 Name KEY

Instructions: Write your work up neatly and attach to this page. Record your final answers (only) directly on this page if they are short; if too long indicate which page of the work the answer is on and mark it clearly. Use exact values unless specifically asked to round.

1. Verify the identity.
   a. $e^x = \sinh x + \cosh x$
   b. $\sinh 2x = 2\sinh x \cosh x$
   c. $\tanh^2 x + \operatorname{sech}^2 x = 1$ pg 2

2. Find the derivative of the function.
   a. $f(x) = \sinh 3x$
   b. $y = \ln\left(\tanh\frac{x}{2}\right)$
   c. $f(t) = \arctan(\sinh t)$
   d. $y = \tanh(3x^2 - 1)$
   e. $g(x) = x\cosh x - \sinh x$ pg 2

3. Find the equation of the tangent line to the function $y = e^{\cosh x}$ at (1,1). pg 2

4. Find the rate of change of the distance between the origin and a moving point on the graph of $y = x^2 + 1$ if $\frac{dx}{dt} = 2\frac{cm}{s}$. pg 2

5. The radius r of a sphere is increasing at a rate of $3\frac{in}{s}$. Find the rate of change of the volume when $r = 9$, and $r = 36$ inches. pg 2

6. All edges of a cube are expanding at a rate of $6\frac{cm}{s}$. How fast is the volume and surface area changing when each edge is 2 cm and again at 10 cm? pg 3

7. A conical tank (with vertex down) is 10 ft across the top and 12 ft deep. If water is flowing into the tank at a rate of $10\frac{ft^3}{min}$, find the rate of change of the depth of the water when the water is 8 ft deep. pg 3

8. A ladder 25 ft long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at $2\frac{ft}{s}$. How fast is the top of the ladder coming down the wall when the base is $7ft, 15ft,$ and 24 ft? pg 3

9. An airplane flies at an altitude of 5 miles toward a point directly over an observer. The speed of the plan is 600 mph. Find the rates of change of the angle of elevation when the angle is $\theta = 30°, \theta = 60°$ and $\theta = 75°$.

pg 3
+30, +90, +111.96 radians per hour → $+\frac{1}{2}$, +1.5, +1.866 radians per minute or +28.65°, +85.94°, +106.91° per min

10. Find any critical points of the function.
    a. $g(x) = x^4 - 4x^2$
    b. $g(t) = 2t\ln t$
    c. $f(x) = \frac{4x}{x^2+1}$
    d. $h(x) = 4x^2(3^x)$ pg 4

11. Locate the absolute extrema of the function on the interval.
    a. $h(x) = -x^2 + 3x - 5, [-2,-1]$
    c. $f(x) = \sqrt[3]{x}, [-1,1]$ pg 4

b. $g(x) = \frac{x^2}{x^2+3}, [-1,1]$ d. $q(x) = \arctan x^2, [-2,1]$

12. Determine whether the mean value theorem can be applied.

a. $y = x^2, [-2,1]$ c. $f(x) = \frac{x+1}{x}, [-1,2]$ pg 5

b. $g(x)e^{-3x}, [0,2]$

$g(x) = e^{-3x}$

13. Apply the mean value theorem to the function $f(x) = x^4 - 8x$ on the interval $[-1,1]$. pg 5

14. Locate any relative extrema and state the intervals on which the function is increasing or decreasing.

a. $f(x) = -2x^2 + 4x + 3$ d. $g(x) = x^4 - 32x + 4$

b. $h(x) = 2x + \frac{1}{x}$ e. $q(x) = \frac{x^2}{x^2-9}$ pg 5

c. $m(x) = x \arctan x$ f. $y = (x-1)e^x$

15. Apply the first derivative test to classify any critical points.

a. $y = \frac{x}{2} + \cos x$ pg 5 b. $f(x) = \cos^2 2x$ pg 6

16. Find any inflection points.

a. $y = -x^4 + 24x^2$ d. $f(x) = x\sqrt{9-x}$

b. $g(x) = \frac{x+1}{\sqrt{x}}$ $\sqrt{x} + \frac{1}{\sqrt{x}}$ e. $h(x) = e^{-\frac{3}{x}}$ pg 6

c. $n(x) = x - \ln x$ f. $p(x) = \arctan x^2$

17. Use the second derivative test to classify the critical points.

a. $f(x) = (x-5)^2$ d. $g(x) = x^4 - 4x^3 + 2$

b. $h(x) = x + \frac{4}{x}$ e. $y = \frac{x}{\ln x}$ pg 6-7

c. $q(x) = x^2e^{-x}$ f. $s(x) = 8x(4^{-x})$

18. Find the differential of the function.

a. $y = 3x^2 - 4$ c. $y = x\sqrt{1-x}$

b. $y = x \cos x$ d. $y = x \arcsin x$ pg 7

19. Use differentials to approximate the value.

a. $\sqrt{99.4}$ b. $\sqrt[3]{28}$ pg 7

MTH 263 Homework #4 Key

1a. $\sinh x + \cosh x = \frac{1}{2}(e^x - e^{-x}) + \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}(e^x \cancel{-e^{-x}} + e^x \cancel{+e^{-x}}) = \frac{1}{2}(2e^x) = e^x$ ✓

b. $\sinh 2x = \dfrac{e^{2x} - e^{-2x}}{2}$

$\cancel{2}\left(\dfrac{e^x - e^{-x}}{\cancel{2}}\right)\left(\dfrac{e^x + e^{-x}}{2}\right) = \frac{1}{2}(e^{2x} - e^{-2x})$ ✓

c. $\tanh^2 x + \text{sech}^2 x = \dfrac{\sinh^2 x}{\cosh^2 x} + \dfrac{1}{\cosh^2 x} = \dfrac{\sinh^2 x + 1}{\cosh^2 x} = \dfrac{\cosh^2 x}{\cosh^2 x} = 1$ ✓

2a. $f'(x) = 3\cosh 3x$

2b. $y' = \dfrac{1}{\tanh \frac{x}{2}} \cdot \text{sech}^2\left(\frac{x}{2}\right) \cdot \frac{1}{2} = \frac{1}{2} \cdot \dfrac{1}{\cosh^{\cancel{2}}\left(\frac{x}{2}\right)} \cdot \dfrac{\cancel{\cosh\left(\frac{x}{2}\right)}}{\sinh\left(\frac{x}{2}\right)} = \dfrac{1}{2\cosh\left(\frac{x}{2}\right)\sinh\left(\frac{x}{2}\right)}$

c. $f'(t) = \dfrac{1}{1 + \sinh^2 t} \cdot \cosh t = \dfrac{1}{\cosh^2 t} \cdot \cosh t = \text{sech}\, t$

d. $y' = \text{sech}^2(3x^2 - 1) \cdot 6x$

e. $g'(x) = \cancel{\cosh x} + x\sinh x - \cancel{\cosh x} = x\sinh x$

3. $y' = e^{\cosh x} \cdot \sinh x \qquad y'\Big|_{(1,1)} = e^{\cosh 1} \cdot \sinh 1 \approx 5.4987\ldots$

$y - 1 = e^{\cosh 1} \cdot \sinh 1\,(x - 1)$

4. $d = \sqrt{x^2 + (x^2+1)^2} = \sqrt{x^2 + x^4 + 2x^2 + 1} = \sqrt{x^4 + 3x^2 + 1}$

$\dfrac{d(d)}{dx} = \frac{1}{2}(x^4 + 3x^2 + 1)^{-1/2}(4x^3 + 6x) = \dfrac{2x^3 + 3x}{\sqrt{x^4 + 3x^2 + 1}}$

$\dfrac{d(d)}{dt} = \dfrac{2x^3 + 3x}{\sqrt{x^4 + 3x^2 + 1}} \dfrac{dx}{dt} \Rightarrow \dfrac{2x^3 + 3y}{\sqrt{x^4 + 3x^2 + 1}}(2)$

5. $V = \frac{4}{3}\pi r^3 \qquad \dfrac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \dfrac{dr}{dt} = 4\pi r^2 \dfrac{dr}{dt} \qquad \dfrac{dr}{dt} = 3$

$\dfrac{dV}{dt}(r=9) = 4\pi(9)^2(3) = 972\pi$ in³/sec

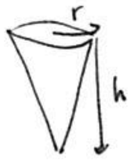
$\dfrac{dV}{dt}(r=36) = 4\pi(36)^2(3) = 15{,}552\pi$ in³/sec

6. $V = s^3$

$\frac{dv}{dt} = 3s^2 \cdot \frac{ds}{dt}$ $\qquad$ $\frac{dv}{dt}(s=2) = 3(2)^2 \cdot 6 = 72 \text{ cm}^3/\text{s}$

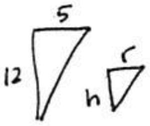
$\frac{dv}{dt}(s=10) = 3(10)^2 \cdot 6 = 1800 \text{ cm}^3/\text{s}$

7.



$V = \frac{1}{3}\pi r^2 h \xrightarrow{\qquad} V = \frac{1}{3}\pi r^2 \left(\frac{12}{5}r\right) =$

$r = 5,\ h = 12$



$\frac{5}{r} = \frac{12}{h}$ $\qquad$ $\frac{4}{15}\pi r^3$

$5h = 12r$

$h = \frac{12}{5}r$ $\qquad$ $\frac{dh}{dt} = \frac{12}{5}\frac{dr}{dt}$

$8 = \frac{12}{5}r$

$\frac{40}{12} = r$ $\qquad$ $\frac{5}{12}\frac{dh}{dt} = \frac{dr}{dt}$

$\frac{dV}{dt} = \frac{4}{15}\pi \cdot 3r^2 \frac{dr}{dt}$

$\frac{4}{5}\pi r^2 \frac{dr}{dt} = \frac{4}{5}\pi\left(\frac{40}{12}\right)^2 \cdot \frac{dr}{dt} = 10$

$\frac{80}{9}\pi \cdot \frac{dr}{dt} = 10$

$\frac{dr}{dt} = \frac{10 \cdot 9}{80\pi} \rightarrow \frac{5}{12}\frac{dh}{dt} = \frac{90}{80\pi} \rightarrow \frac{dh}{dt} = \frac{27}{10\pi}$ ft/min

8.

$x^2 + y^2 = 25$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ $\qquad$ $2y\frac{dy}{dt} = -2x\frac{dx}{dt} \rightarrow \frac{dy}{dt} = \frac{-2x}{2y}\frac{dx}{dt}$

$\frac{dx}{dt} = 2$ ft/sec.

$x = 7 \quad y = 24$

$\frac{dy}{dt} = -\frac{7}{24}(2) = \frac{-7}{12}$ ft/sec

$\frac{dy}{dt}$ $x = 15,\ y = 20$ $= -\frac{15}{20}(2) = \frac{-15}{10}$ ft/sec

$\frac{dy}{dt}$ $x = 24 \ y = 7$ $= \frac{-24}{7}(2) = -\frac{48}{7}$ ft/sec

$7^2 + y^2 = 25^2 \rightarrow y^2 = 576$

$y = 24$

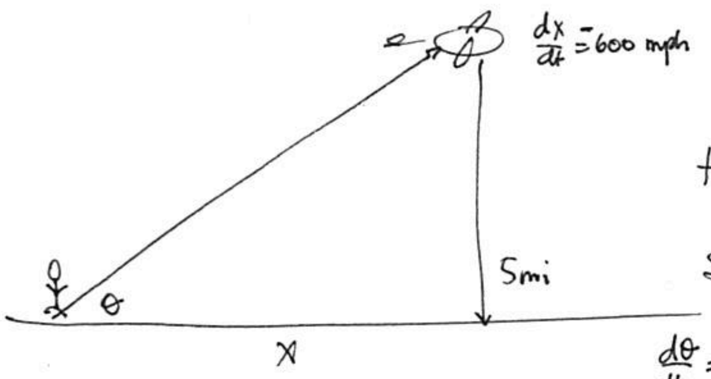
$15^2 + y^2 = 25^2 \rightarrow y^2 = 400$

$y = 20$

$24^2 + y^2 = 25^2 \rightarrow y^2 = 49$

$y = 7$

9.



$\frac{dx}{dt} = 600$ mph

$x \tan\theta = y$

$x = \frac{y}{\tan\theta} = \frac{5}{\tan\theta}$

$\tan\theta = \frac{y}{x}$ $\qquad$ $y = 5$

$\frac{dy}{dt} = 0$

$\sec^2\theta \frac{d\theta}{dt} = \cancel{\frac{dy}{dt}}^{\,0} \cdot \frac{1}{x} - \frac{y}{x^2}\frac{dx}{dt}$

$\frac{d\theta}{dt} = \cos^2\theta \left[-\frac{5}{x^2} \cdot -600 \text{ mph}\right]$ $\qquad$ $\frac{d\theta}{dt} = +$[illegible]90

$\theta = 30° \rightarrow x = \frac{5}{\sqrt{3}}$ $\quad$ $\theta = 60° \rightarrow x = 5\sqrt{3}$ $\quad$ $\theta = 75° \rightarrow x = 1.339746$ $\quad$ $\frac{d\theta}{dt} = \frac{3}{4}\left[\frac{-5}{(5/\sqrt{3})^2} \cdot -600\right] = +30$ $\quad$ $\frac{d\theta}{dt} = +111.96$

10 a. $g'(x) = 4x^3 - 8x$

$4x(x^2-2) = 0$

$x=0,\ x = \pm\sqrt{2}$

b. $g'(t) = 2\ln t + \frac{2\cancel{t}}{\cancel{t}} = 2\ln t + 2$

$0 = 2\ln t + 2$

$\ln t = -1$

$t = \frac{1}{e}$

c. $f'(x) = \frac{4(x^2+1) - 2x(4x)}{(x^2+1)^2} = \frac{4x^2+4-8x^2}{(x^2+1)^2} = \frac{4-4x^2}{(x^2+1)^2} = 0$ $\quad 4-4x^2=0$

$4x^2 = 4$

$x^2 = 1 \quad x = \pm 1$

d. $h'(x) = 8x(3^x) + 4x^2 \cdot \ln 3 \cdot 3^x$

$0 = 3^x(8x + 4x^2\ln 3) = 3^x \cdot x(8 + 4\ln 3 \cdot x)$

$x = 0,\ 8 = -4\ln 3\, x \rightarrow \frac{8}{-4\ln 3} = x \Rightarrow \frac{-2}{\ln 3} = x$

11. a. $h'(x) = -2x + 3$

$x = \frac{3}{2}$ not on interval

$h(-2) = -15$ ← abs min

$h(-1) = -9$ ← abs max

b. $g'(x) = \frac{2x(x^2+3) - 2x(x^2)}{(x^2+3)^2} = \frac{2x^3+6x-2x^3}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2} \rightarrow 6x = 0$

$x = 0$ on interval

$g(0) = 0$ ← abs. min

$g(-1) = 1/4$ ← abs max

$g(1) = 1/4$ ← abs max

c. $f'(x) = \frac{1}{3}x^{-2/3}$ $\quad x=0$ on interval

$f(-1) = -1$ ← abs. min

$f(0) = 0$

$f(1) = +1$ ← abs max

d. $f'(x) = \frac{2x}{1+x^4} = 0$ $\quad x=0$ on interval

$f(-2) = 1.3258$ ← abs max

$f(0) = 0$ ← abs min

$f(1) = 0.7854$

12a.

yes. the function is continuous on the interval

b. yes, the function is continuous on the interval

c. no. The function is not continuous on the interval

13. $f(-1) = (-1)^4 - 8(-1) = 1+8=9$

$f(1) = (1)^4 - 8(1) = 1-8 = -7$

$m = \frac{-7-9}{1-(-1)} = \frac{-16}{2} = -8$

$f'(x) = 4x^3 - 8 =$

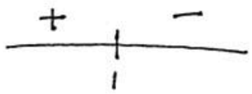
$4x^3 - \cancel{8} = \cancel{-}8$

$4x^3 = 0 \rightarrow \underline{\underline{x=0}}$ $f'(0) = -8$ ✓

14.a. $f'(x) = -4x + 4 = 0$

$4x = 4$

$x = 1$



increasing on $(-\infty, 1)$

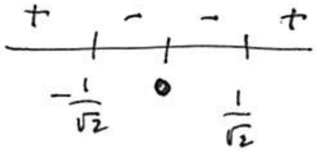
decreasing on $(1, \infty)$

b. $h'(x) = 2 - \frac{1}{x^2} = 0$

$\frac{1}{x^2} = 2$

$x^2 = \frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}$

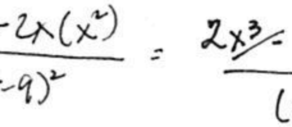


increasing on $(-\infty, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, \infty)$

decreasing on $(-\frac{1}{\sqrt{2}}, 0) \cup (0, 1/\sqrt{2})$

c. $m'(x) = \arctan x + \frac{x}{1+x^2} = 0$

$x \approx -0.747212$



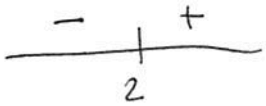
increasing on $(-0.747212, \infty)$

decreasing on $(-\infty, -0.747212)$

d. $g'(x) = 4x^3 - 32 = 0$

$4(x^3 - 8) = 0$

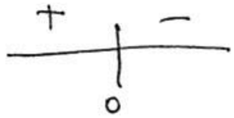
$x = 2$



increasing on $(2, \infty)$

decreasing on $(-\infty, 2)$

e. $q(x) = \frac{x^2}{x^2-9}$ $q'(x) = \frac{2x(x^2-9) - 2x(x^2)}{(x^2-9)^2} = \frac{\cancel{2x^3} - 18x - \cancel{2x^3}}{(x^2-9)^2} = \frac{-18x}{(x^2-9)^2}$ $x=0$

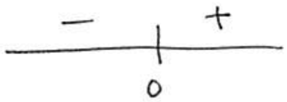


increasing on $(-\infty, 0)$

decreasing on $(0, \infty)$

f. $y' = e^x + (x-1)e^x = xe^x$

$x=0$



increasing on $(0, \infty)$

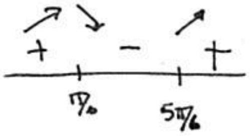
decreasing on $(-\infty, 0)$

15a. $y' = \frac{1}{2} - \sin x = 0$

$\sin x = \frac{1}{2}$

$x = \pi/6, 5\pi/6$

$\pm 2\pi$

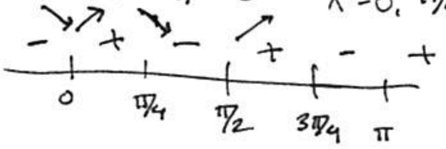


max at π/6

min at 5π/6

15b. $f'(x) = -2\cos 2x \cdot \sin 2x \cdot 2 = -4\cos 2x \cdot \sin 2x = 0$

$-2\sin 4x = 0$ $\quad x = 0, \pi/4, \pi/2, 3\pi/4$, etc.



$0, \pi/2, \pi$, etc are minima

$\pi/4, 3\pi/4$, etc are maxima

16a. $y' = -4x^3 + 48x$

$y'' = -12x^2 + 48 = 0$

$12x^2 = 48$

$x^2 = 4$

$x = \pm 2$ inflection

16b. $g'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = 0$

$x^{-1/2} = x^{-3/2} \rightarrow x^{1/2} = x^{3/2} \rightarrow (x^{3/2} - x^{1/2}) = 0$

$x^{1/2}(x - 1) = 0$ $\quad$ critical $x = 1$, $x = 0$ $\quad$ inflection $x = 0$, $x = 1/3$

$g''(x) = -\frac{1}{4}x^{-3/2} + \frac{3}{4}x^{-5/2} = -1/4\,(x^{-5/2})(1 - 3x) = 0$

16. c. $h'(x) = 1 - \frac{1}{x} = 0$

$1 = \frac{1}{x}$ $\quad x = 1$ critical

$h''(0) = +\frac{1}{x^2} \rightarrow x = 0$ inflection

d. $f'(x) = \sqrt{9-x} + x \cdot \frac{1}{2}(9-x)^{-1/2}(-1) =$

$f''(x) = \frac{1}{2}(9-x)^{-1/2}(-1) - \frac{1}{2}(9-x)^{-1/2} - x \cdot \frac{1}{2} \cdot (-\frac{1}{2})(9-x)^{-3/2}(-1)$

$\frac{1}{2}(9-x)^{-3/2}\left[(9-x)(-1) - (9-x) - x(1/2)\right] =$

$\frac{1}{2}(9-x)^{-3/2}\left[-9 + x - 9 + x - \frac{1}{2}x\right] =$

$\left[\frac{3}{2}x - 18\right]$ $\quad x = 9$ inflection

$\frac{3}{2}x = 18 \rightarrow x = 18 \cdot \frac{2}{3} = 12$

e. $h'(x) = e^{-3/x} \cdot \left(\frac{3}{x^2}\right)$

$h''(x) = e^{-3/x}\left(\frac{3}{x^2}\right)^2 + e^{-3/x}\left(\frac{-6}{x^3}\right) \cdot$

$e^{-3/x}\left[\frac{9}{x^4} - \frac{6}{x^3}\right] = 0$

$9x - 6 = 0$ $\quad x = \frac{6}{9} = \frac{2}{3}$

$x = 0$ inflection

f. $p'(x) = \frac{2x}{1+x^4}$

$p''(x) = \frac{2(1+x^4) - 4x^3(2x)}{(1+x^4)^2} = \frac{2 + 2x^4 - 8x^4}{(1+x^4)^2} =$

$\frac{2 - 6x^4}{(1+x^4)^2} = 0$

$2 - 6x^4 = 0$

$6x^4 = 2$

$x^4 = \frac{1}{3}$ $\quad x = \pm\frac{1}{\sqrt[4]{3}}$ inflection points.

17a. $f'(x) = 2(x-5) = 0$ $\quad x = 5$

$f''(x) = 2$ $\quad$ min. since $f'' > 0$

b. $h'(x) = 1 - \frac{4}{x^2} = 0$ $\quad \frac{4}{x^2} = 1 \rightarrow x^2 = 4$

$x = \pm 2$

$h''(x) = \frac{8}{x^3}$ $\quad$ inf. at $x = 0$ $\quad x = -2$ $f''(x) < 0$ max

$x = 2$ $f''(x) > 0$ min

c. $g'(x) = 2xe^{-x} - x^2e^{-x} = e^{-x}(2x - x^2) = 0$ $\quad x = 0, x = 2$

$g''(x) = e^{-x}(-1)(2x - x^2) + e^{-x}(2 - 2x) = e^{-x}(x^2 - 2x + 2 - 2x) = e^{-x}(x^2 - 4x + 2)$

$x = 0$ $f''(x) > 0$ min

$x = 2$ $f''(x) < 0$ max

17 d. $g'(x) = 4x^3 - 12x^2 \quad 4x^2(x-3) \quad x=0,\ x=3$

$g''(x) = 12x^2 - 24x = 0 \quad 12x(x-2)$

$x=0,\ x=2$ inflection

$x=0$ cannot be determined; $x=3,\ f''(x)>0$ min

e. $y' = \frac{\ln x - x \cdot \frac{1}{x}}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} = \frac{1}{\ln x} - \frac{1}{(\ln x)^2} = (\ln x)^{-1} - (\ln x)^{-2}$

$y'' = -1(\ln x)^{-2} - (-2)(\ln x)^{-3} = 0 \quad \frac{-\ln x + 2}{(\ln x)^3} = 0 \quad \ln x = 2$

$x = e^2$ inflection

$(\ln x)^{-1} - (\ln x)^{-2} = 0$

$x = e,\ f''(x) > 0$ min

$\ln x - 1 = 0 \quad x = e$

f. $S'(x) = 8(4^{-x}) + 8x(-1)(\ln 4)4^{-x} = 4^{-x}(8 - 8\ln 4 \cdot x) = 0 \quad \ln 4x = 1$

$x = \frac{1}{\ln 4}$

$S''(x) = (-1)\ln 4 \cdot 4^{-x}(8 - 8\ln 4 \cdot x) + 4^{-x}(-8\ln 4) = 0$

$4^{-x}[-\ln 4 \cdot 8 + 8(\ln 4)^2 x - 8\ln 4] = 0 \quad 8(\ln 4)^2 x = 16\ln 4$

$x = \frac{2\cancel{16\ln 4}}{8(\ln 4)^2} = \frac{2}{\ln 4}$

$x = 1/\ln 4,\ f''(x) < 0$ max

18a. $dy = (6x-4)dx$

b. $dy = [\cos x - x\sin x]dx$

c. $dy = \left[\sqrt{1-x} + x \cdot \frac{1}{2}(1-x)^{-1/2}\right]dx$

d. $dy = \left[\arcsin x + x \cdot \frac{1}{\sqrt{1-x^2}}\right]dx$

19a. $f(x) = \sqrt{x} \quad x = 100 \quad \Delta x = -0.6$

$f'(x) = \frac{1}{2\sqrt{x}}$

$dy = \frac{1}{2\sqrt{100}}(-0.6) = \frac{6}{2 \cdot 10 \cdot 10} = \frac{3}{100} = -0.03$

$y + dy \approx \sqrt{100} - 0.03 =$ ~~10.03~~ 9.97

b. $f(x) = \sqrt[3]{x} \quad x = 27 \quad dx = 1$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$

$dy = \frac{1}{3(\sqrt[3]{27})^2}(1) = \frac{1}{3 \cdot (3)^2} = \frac{1}{27}$

$y + dy = 3 + \frac{1}{27} \approx 3.\overline{037}$