6/1/2023

Continuity
Formal Definition of a Limit – Proofs
Definition of a Derivative

Continuity (2.4)

Concept: can you draw the function without picking up your pencil?

Continuity requires three properties to be true:
1) The two one-sided limits exist (and not infinite)
2) The two one-sided limits have to agree – so that the two-sided limit exists (not infinite)
3) The value of the function and the value of the limit are the same

1) $\lim_{x \to c^-} f(x) = L_1$ and $\lim_{x \to c^+} f(x) = L_2$
2) $L_1 = L_2, \lim_{x \to c} f(x) = L$
3) $\lim_{x \to c} f(x) = f(c)$

Discontinuities come in three flavors:

1) Hole/point discontinuity (the graph is continuous except at a single value, and the hole can be repaired by essentially replacing it with one number), removable discontinuities

2) Jump discontinuity – the break in the graph is separating the pieces by a vertical distance, but both one-sided limits are finite. Non-removable discontinuity

3) Infinite discontinuity – when there is a vertical asymptote

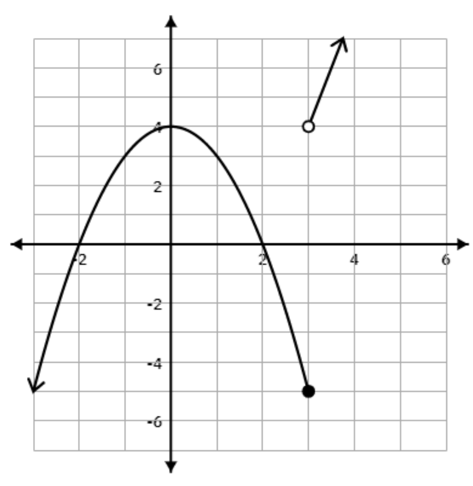
Example.

$$f(x) = \begin{cases} -x^2 + 4, x \leq 3 \\ 4x - 8, x > 3 \end{cases}$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} -x^2 + 4 = -5$$

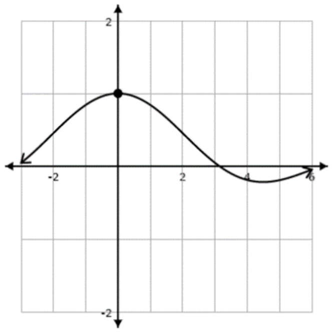
$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 4x - 8 = 4$$

These limits are not equal to each other. So the $\lim_{x \to 3} f(x) = DNE$.
The graph is not continuous at x=3. This is a jump discontinuity, and it is not removable.



Example.

$$f(x) = \begin{cases} \frac{\sin(x)}{x}, x \neq 0 \\ 1, \quad x = 0 \end{cases}$$



The second piece here is to repair the removable discontinuity and now the new function is continuous. $f(x)$ is continuous, even though $\frac{\sin(x)}{x}$ is not.

Formal $(\varepsilon - \delta)$ definition of a limit:

$f(x)$ is a function defined on an open interval, except possibly at $x = a$, but a is inside the interval. And L is a real number.

Then we say that

$$\lim_{x \to a} f(x) = L$$

If for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Example.

Use the epsilon-delta definition of a limit to prove that $\lim_{x \to 1}(2x + 1) = 3$.

Work backwards to get the relationships that I need for the proof.

$$|(2x + 1) - 3| < \varepsilon$$

$$|2x - 2| < \varepsilon$$

$$2|x - 1| < \varepsilon$$

$$|x - 1| < \frac{\varepsilon}{2} = \delta$$

This is the relationship that I need to make the proof work… but it is not the proof.

Proof:
Suppose that $|x - 1| < \delta$. And let $\delta = \frac{\varepsilon}{2}$. Then, $|x - 1| < \frac{\varepsilon}{2} \rightarrow 2|x - 1| < \varepsilon \rightarrow |2x - 2| < \varepsilon$

$$\rightarrow |(2x + 1) - 3| < \varepsilon$$

Therefore, the $\lim_{x \to 1}(2x + 1) = 3$.
QED.

Example.
Use the epsilon-delta definition of a limit to prove that $\lim_{x \to -1}(x^2 - 4) = -3$.

Work to get to the relationship between epsilon and delta:

$$|(x^2 - 4) - (-3)| < \varepsilon$$

$$|x^2 - 1| < \varepsilon$$

Goal: get to $|x + 1| < \delta$

$$|x - 1| \cdot |x + 1| < \varepsilon$$

Suppose that being "near" x=-1 means that I'm within one unit of x=-1, so I'm on the interval (-2,0). Where is the value of |x-1| a maximum on this interval?

|-2-1|=|-3|=3
|0-1|=|-1|=1

On the interval (-2,0) near x=-1, the expression |x-1|<3 for all values of x on this interval.

Therefore:

$$|x - 1| \cdot |x + 1| < \varepsilon$$

Implies

$$3|x + 1| < \varepsilon$$

$$|x + 1| < \frac{\varepsilon}{3} = \delta$$

Proof:

Suppose that $|x + 1| < \delta$, and further suppose that $\delta = \frac{\varepsilon}{3}$. Then we can say that

$$|x + 1| < \delta \to |x + 1| < \frac{\varepsilon}{3} \to 3|x + 1| < \varepsilon$$

If $\delta < 1$, then we are on the interval $(-2,0)$, and the expression $|x - 1| < 3$.

$$3|x + 1| < \varepsilon \to |x - 1| \cdot |x + 1| < \varepsilon \to |x^2 - 1| < \varepsilon \to |(x^2 - 4) + 3| < \varepsilon \to$$

$$|(x^2 - 4) - (-3)| < \varepsilon$$

Therefore: $\lim_{x \to -1}(x^2 - 4) = -3$.
QED.

The formal definition of the derivative (limit definition). (3.1)

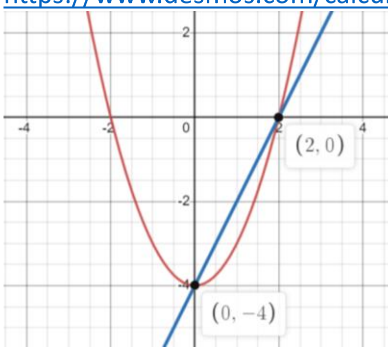
$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

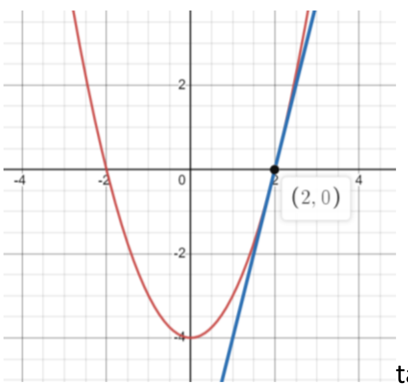
Without the limit, the slope formula $\frac{f(x+h)-f(x)}{h}$, is called the difference quotient. The slope of the secant line.

The slope of the tangent line touches the graph at only one point, and represents the slope of the graph at that one point.

https://www.desmos.com/calculator



secant line



tangent line only touches the graph at a single point.

The derivative is a function that can provide us the value of the slope of the tangent at any point on the curve.

Example.
Find the derivative using the limit definition for $f(x) = 3x + 4$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{3(x+h) + 4 - (3x+4)}{h} = \lim_{h \to 0} \frac{3x + 3h + 4 - 3x - 4}{h} = \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3$$

Any pair of points on a straight line has the same slope, so the tangent line (as those two points get closer together) also has the be the same value.

Example.
Find the derivative using the limit definition for $f(x) = x^2 - 4$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - 4 - (x^2 - 4)}{h}$$

$$(x+h)^2 = x^2 + 2xh + h^2$$
$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

$$\lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 4 - x^2 + 4}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} - \lim_{h \to 0} 2x + h = 2x$$

$$f'(x) = 2x$$

What is the slope of the tangent line at x=2? f'(2)=2(2)=4
What is the slope of the tangent line at x=1? f'(1)=2(1)=2
What is the slope of the tangent line at x=0? f'(0)=2(0)=0

When the function involves a rational function, like 1/x, find a common denominator. Then things will cancel, and you will get to a point where only h terms remain and allow for cancelling the h in the denominator.
When the function involves a radical, you will need to multiply by the conjugate to the radicals out of the numerator… so that that terms will cancel and leave you with an h you can use to eliminate the h in the denominator.

Example.
Find the derivative using the limit definition for $f(x) = \frac{1}{x}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h} \cdot \frac{x}{x} - \frac{1}{x} \cdot \frac{x+h}{x+h}\right) =$$

$$\lim_{h \to 0} \frac{1}{h}\left(\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}\right) = \lim_{h \to 0} \frac{1}{h}\left(\frac{x - (x+h)}{x(x+h)}\right) = \lim_{h \to 0} \frac{1}{h}\left(\frac{x - x - h}{x(x+h)}\right) =$$

$$\lim_{h \to 0} \frac{1}{h}\left(\frac{-h}{x(x+h)}\right) = \lim_{h \to 0} \left(\frac{-1}{x(x+h)}\right) = -\frac{1}{x^2}$$

On the video playlist is an example of a rational function using the definition.