6/6/2023

Derivative as a Function – continuity/differentiability, graphs of functions vs. derivatives, higher derivatives
Derivative Rules – power rule, product/quotient rules, horizontal tangents
Derivative as Rates of Change – velocity, acceleration, marginal cost/revenue

Last time we defined the derivative formally as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

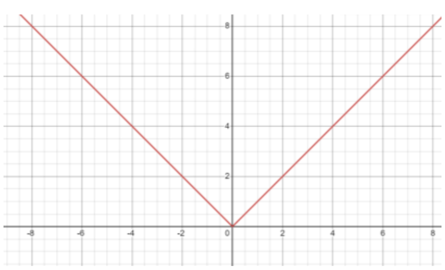
Notation
The derivative:
The first derivative: $f'(x)$ “read as f-prime of x”
Leibniz notation: $\frac{df}{dx}$ more important in working with implicit function (relations), with related rates
Physics: $\dot{x}(t)$ means that derivative is being taken with respect to time.
Mathier notations: $D_x(f)$ (operator notation)

The relationship between continuity of a function and differentiability

If a function is differentiable (the derivative is defined everywhere in the domain), then the function is continuous.

This does not go in the other direction: it’s not the case that every continuous function has a continuous derivative (it’s not differentiable).

The typical example is the absolute value function, but also is true of many continuous, piecewise defined functions.



$$f(x) = |x| = \begin{cases} -x, x < 0 \\ x, \quad x \geq 0 \end{cases}$$

$$\lim_{\Delta x \to 0^-} \frac{-(x+\Delta x) - (-x)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-x - \Delta x + x}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1$$

$$\lim_{\Delta x \to 0^+} \frac{(x+\Delta x) - (x)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{x + \Delta x - x}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{\Delta x}{\Delta x} = 1$$

$$\lim_{\Delta x \to 0} \frac{|x+\Delta x| - |x|}{\Delta x} = DNE$$

This function is not continuously differentiable, but it is continuous.

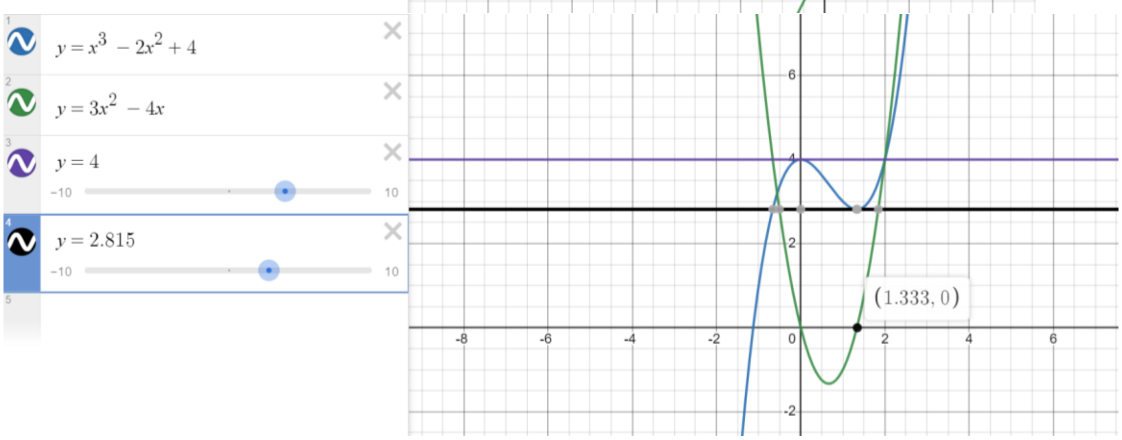
Cusps (points where a function, or pieces of a function, meet at a point) will not be differentiable.

You may be given problems with a piecewise defined function where you are asked to find the values of constants that will make the function differentiable (and continuous).

$$f(x) = \begin{cases} x^2 - 1, x < 2 \\ ax + b, x \geq 2 \end{cases}$$

To make this work: set the two pieces equal to each other when x=2, and set their derivatives equal to each other when x=2.

The relationship between graphs and their derivatives.



Higher order derivatives:

If you take a derivative of $f(x)$, then you get $f'(x)$. What happens if you take the derivative of $f'(x)$? The second derivative $f''(x)$ read "f-double-prime of x".

Leibniz notation for the second derivative is $\frac{d^2 f}{dx^2}$. The operator for the derivative is written as $\frac{d}{dx}$ "take the derivative with respect to x". $\frac{d}{dx}[f(x)] = \frac{df}{dx}$, but for the second derivative $\frac{d}{dx}\left[\frac{df}{dx}\right] = \frac{d^2}{dx^2}[f(x)]$.

Likewise, there is a third derivative: the derivative of the second derivative.

$$f'''(x) = \frac{d^3 f}{dx^3}$$

The prime notation takes a bit of a turn after this:
The fourth derivative:

$$f^{IV}(x) = f^{(4)}(x)$$

The parentheses around the number are required to separate it from the fourth power.

If the first derivative is velocity, then the second derivative is acceleration. The third derivative is the jerk.

Short-cut Rules for derivatives.
The power rule:
Apply to any function we can write as $f(x) = x^n$ where n is any real number.

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

$$x^2: \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} = \frac{2x + h}{1}$$

$$x^3: \frac{(x+h)^3 - x^3}{h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \frac{3x^2h + 3xh^2 + h^3}{h} = \frac{3x^2 + 3xh + h^2}{1}$$

$$\frac{1}{x} = x^{-1}: -\frac{1}{x^2}$$

$$\sqrt{x} = x^{1/2}: \frac{1}{2\sqrt{x}}$$

Properties of derivatives:
If the function is a constant $f(x) = c$, the derivative is 0: $\frac{d}{dx}[c] = 0$
If the function is sum or difference of terms, then we can take the derivative term by term:
$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$

Scalar multiples can be pulled out: $\frac{d}{dx}[kf(x)] = k\frac{d}{dx}[f(x)]$

Take the derivative of $f(x) = x^4 - 3x^3 + 7x - 11$ (using the short-cut rule/power rule).

$$f'(x) = \frac{d}{dx}[x^4] - \frac{d}{dx}[3x^3] + \frac{d}{dx}[7x] - \frac{d}{dx}[11]$$

$$= \frac{d}{dx}[x^4] - 3\frac{d}{dx}[x^3] + 7\frac{d}{dx}[x] - \frac{d}{dx}[11]$$

$$4x^3 - 3(3x^2) + 7(1x^0) - 0$$

$$= 4x^3 - 9x^2 + 7$$

Find the derivative of $f(x) = \frac{1}{x^2} - \sqrt[3]{x} + 6\sqrt{x^5} + x^e = x^{-2} - x^{1/3} + 6x^{5/2} + x^e$

(Note: $\frac{1}{3x^2} = \frac{1}{3}x^{-2}$)

$$f'(x) = -2x^{-3} - \frac{1}{3}x^{-2/3} + 6\left(\frac{5}{2}\right)x^{3/2} + ex^{e-1}$$

$$f'(x) = -\frac{2}{x^3} - \frac{1}{3\sqrt[3]{x^2}} + 15\sqrt{x^3} + ex^{e-1}$$

Product Rule:
Use this rule when our function is the product of two simpler functions.

$$h(x) = f(x)g(x)$$

$$h(x) = (x^3 - 5x^2)\left(x^2 - \frac{1}{x} + \sqrt{x}\right) = (x^3 - 5x^2)\left(x^2 - x^{-1} + x^{\frac{1}{2}}\right)$$

$$h'(x) = f'(x)g(x) + g'(x)f(x)$$

What happens if I FOIL? Leading term is x^5 and so its derivative is $5x^4$… and then a bunch of other terms.

What happens if I use $f'(x)g'(x)$? the leading term of $f'(x) = 3x^2$, and the leading term of $g'(x) = 2x$, and so the leading term of the product is $6x^3$…. Which is not $5x^4$.

$$h(x) = (x^3 - 5x^2)\left(x^2 - \frac{1}{x} + \sqrt{x}\right) = (x^3 - 5x^2)\left(x^2 - x^{-1} + x^{\frac{1}{2}}\right)$$

$f(x) = x^3 - 5x^2$	$g(x) = x^2 - x^{-1} + x^{\frac{1}{2}}$
$f'(x) = 3x^2 - 10x$	$g'(x) = 2x + x^{-2} + \frac{1}{2}x^{-\frac{1}{2}}$

$$h'(x) = (3x^2 - 10x)\left(x^2 - x^{-1} + x^{\frac{1}{2}}\right) + \left(2x + x^{-2} + \frac{1}{2}x^{-\frac{1}{2}}\right)(x^3 - 5x^2)$$

Quotient Rule

$$h(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$$

$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

$$h(x) = \frac{2x - 5}{x^2 + 1}$$

$f(x) = 2x - 5$	$g(x) = x^2 + 1$
$f'(x) = 2$	$g'(x) = 2x$

$$h'(x) = \frac{2(x^2 + 1) - 2x(2x - 5)}{(x^2 + 1)^2}$$

Combining rules

$$m(x) = \frac{f(x)g(x)}{h(x)}$$

$$m(x) = \frac{(x^2 + 4)(x^3 - x^{-4})}{5x - 7}$$

$(x^2 + 4)(x^3 - x^{-4})$	$5x - 7$
$(x^2 + 4)(3x^2 + 4x^{-5}) + (2x)(x^3 - x^{-4})$	5

$x^2 + 4$	$x^3 - x^{-4}$
$2x$	$3x^2 + 4x^{-5}$

$$m'(x) = \frac{[(x^2 + 4)(3x^2 + 4x^{-5}) + (2x)(x^3 - x^{-4})](5x - 7) - 5(x^2 + 4)(x^3 - x^{-4})}{(5x - 7)^2}$$

Derivatives represent functions that are rates of change

Physics, if the function represents the position of a particle along a path, then the first derivative is the velocity of the particle at a point on that point (speed is the absolute value of the velocity). The second derivative is the rate of change of the velocity: acceleration.

In business, the derivative is referred as the “marginal” something or other. The rate of change of the cost is the marginal cost. The rate of change of the revenue is the marginal revenue. The rate of change of the profit is the marginal profit.