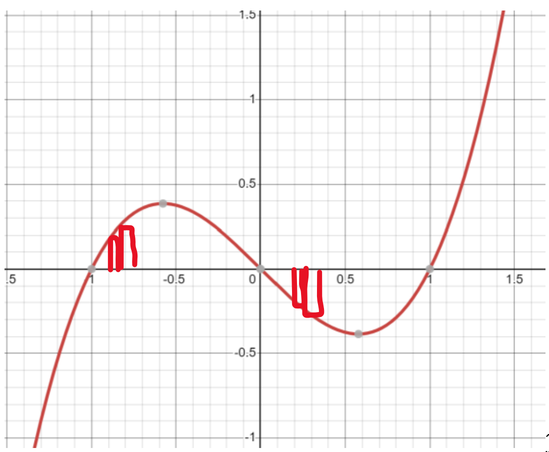
7/13/2023

Come back to any missing bits
Substitution
Change of variables

Properties of integrals

1) Splitting the integral in the middle of the interval
2) Even symmetry
3) Odd symmetry

1. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
   Mostly applied when we are looking for geometric area and need the signs to be positive.
   If the functions cross more than twice.

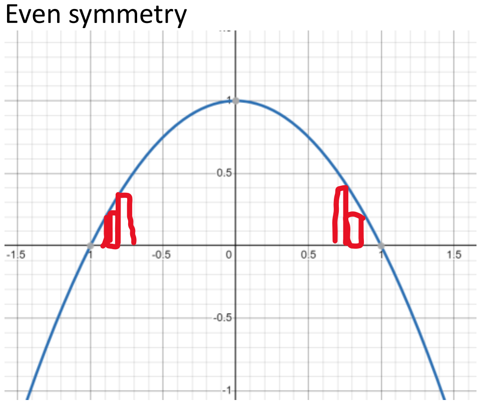


$y = x(x+1)(x-1) = x(x^2-1) = x^3 - x$

We would want to split this integral into two pieces. One from [-1,0] where the area is already positive. And another from [0,1] where the area is negative. To get the geometric area, we would take the absolute value of both regions and then add them.

3. Odd symmetry can be seen from the same graph. The reason odd functions cancel is because the positive area is the same size as the negative area.

2. Even symmetry



$y = 1 - x^2$

The area between [-1,0] is exactly the same value as the area between [0,1]

Example using properties of symmetry

$$\int_{-2}^{2} \sin(x) + x^2 - x^3 + 4 - \frac{1}{x} + 6x^{2/3} dx =$$

$$\int_{-2}^{2} \sin(x) - x^3 - \frac{1}{x} dx + \int_{-2}^{2} x^2 + 4 + 6x^{2/3} dx =$$

$$0 + 2\int_{0}^{2} x^2 + 4 + 6x^{2/3} dx =$$

$$2\left[\frac{x^3}{3} + 4x + 6\left(\frac{3}{5}\right)x^{\frac{5}{3}}\right]_0^2 = 2\left[\frac{8}{3} + 8 + \frac{18}{5}\left(2^{\frac{5}{3}}\right)\right]$$

Substitution

(u-substitution) we are trying to undo the chain rule – composite function (one with a composition $f\big(g(x)\big)$ and often $g'(x)$.

Recall from the chain rule:

$$\frac{d}{dx}[\sin x^2] = \cos x^2 \ \ (2x) = 2x \cos x^2$$

Reverse this process?

$$\int 2x \cos x^2 \, dx$$

Identifying the composite function.
Is the function multiplying this function, is it the derivative (or a constant multiple of that derivative) of the inside function in the composite function?
If the answer to that question is yes, you can use substitution.

Compare to $\int 2x \cos x \, dx$
There are no composite functions here.
Compare to $\int 2x \cos 2x \, dx$
The product, 2x on the outside, is NOT the derivative of the inside function.

$$\frac{d}{dx}[2x] \neq 2x$$

$$\int 2x \cos x^2 \, dx$$

The composite function is $\cos x^2$, the inside function is x^2, and the derivative of x^2 is 2x which is the function sitting out front. (any multiple of x will do).

Make a u-substitution: $u = x^2$ (the inside function.

$$du = 2xdx$$

Do the substitution

$$\int 2x \cos x^2 \, dx = \int (\cos x^2) 2xdx = \int \cos u \, du = \sin u + C = \sin x^2 + C$$

$$\int \frac{e^x}{e^x + 1} dx =$$

When there is a fraction in the integral (a ratio of functions), is the derivative of the denominator in the numerator? (give or take a constant multiple)

$$\frac{d}{dx}[e^x + 1] = e^x$$

Let the denominator be u

$$u = e^x + 1, du = e^x dx$$

$$\int \frac{e^x}{e^x + 1} dx = \int \frac{1}{u} du = \ln|u| + C = \ln|e^x + 1| + C$$

Example.

$$\int \frac{x}{x^2 + 1} dx$$

$$let\ u = x^2 + 1, du = 2xdx \rightarrow \frac{1}{2} du = xdx$$

$$\int \frac{x}{x^2 + 1} dx = \int \frac{1}{2} \left(\frac{1}{u}\right) du = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln|x^2 + 1| + C$$

Example.

$$\int x^2 \sec x^3 \tan x^3 \, dx$$

Composite function: $\sec x^3 \tan x^3$, they have the same inside function, x^3.

Let $u = x^3$, $du = 3x^2 dx \rightarrow \frac{1}{3} du = x^2 dx$

$$\int x^2 \sec x^3 \tan x^3 \, dx = \int (\sec x^3 \tan x^3)\, x^2 dx = \int \frac{1}{3} \sec u \tan u \, du = \frac{1}{3} \sec u + C = \frac{1}{3} \sec x^3 + C$$

Example.

$\int \sin x \cos x \, dx$ or $\int \sec^2 x \tan x \, dx$

In these cases, one of the functions is actually the derivative of the other function.

In the sin(x)cos(x) case, I'm going to pick $u = \sin x\,, du = \cos x\, dx$

$$\int \sin x \cos x\, dx = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$$

$$\int \sec^2 x \tan x\, dx$$

Let $u = \tan x\,, du = \sec^2 x\, dx$

$$\int \sec^2 x \tan x\, dx = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C$$

$$\int \sec^2 x \tan x\, dx = \int (\sec x) \sec x \tan x\, dx$$

Change of Variable

Is a method of integrating problems that are not the result of chain rule, but some of the techniques (notation) are similar.

$$\int x\sqrt{x+1}dx$$

The composite function $\sqrt{x+1}$, but $\frac{d}{dx}[x+1] \neq x$.

Not a u-substitution.

$$u = \sqrt{x+1}$$

Use algebra to solve for the rest of the expression (here, x)

$$u^2 = x + 1$$
$$u^2 - 1 = x$$

And then take the derivative

$$2udu = dx$$

Then substitute everything into the integral

$$\int x\sqrt{x+1}dx = \int (u^2-1)u(2udu) = \int 2u^2(u^2-1)du = \int 2u^4 - 2u^2 du = \frac{2}{5}u^5 - \frac{2}{3}u^3 + C =$$

$$\left(\frac{2}{5}\right)(x+1)^{\frac{5}{2}} + \frac{2}{3}(x+1)^{\frac{3}{2}} + C$$

$$\int x\sqrt{x^2+1}\,dx$$

This is traditional substitution

$$\int x^2 \sqrt[3]{2x - 1}dx$$

Use change of variables because the derivative of 2x-1 is not x^2

Try traditional substitution first. If it won't work, then try change of variables.

For next week: review factoring, review completing the square