7/18/2023

Integrals involving inverse trig functions/exponential and log functions
Hyperbolic trig functions

$$\int e^x dx = e^x + C$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C$$

$$\int a^x dx = \frac{1}{\ln a} a^x + C$$

$$\int \frac{1}{x} dx = \int x^{-1} dx = \ln x + C$$

$$\log_a x = \frac{\ln x}{\ln a}$$

$$\int \ln x \, dx = x \ln x - x + C$$

Example.

$$\int 2^x dx = \frac{1}{\ln 2} 2^x + C$$

$$\int 2^x dx = \int e^{(\ln 2)x} dx = \frac{1}{\ln 2} e^{(\ln 2)x} + C = \frac{1}{\ln 2} 2^x + C$$

Example.

$$\int \frac{1}{x \ln x} dx$$

$$u = \ln x \, , du = \frac{1}{x} dx$$

$$\int \frac{1}{\ln x} \left(\frac{1}{x} dx \right) = \int \frac{1}{u} du = \ln u + C = \ln|\ln x| + C$$

Example.

$$\int \frac{x+1}{x^2 + 2x + 6} dx$$

1) Do I need to do long division? If the numerator is the same degree or larger than the denominator, then yes. Otherwise, no.
2) Is it u-substitution? Let u be the denominator and see if the numerator is a multiple of the derivative.

$$u = x^2 + 2x + 6, du = (2x + 2)dx = 2(x + 1)dx \rightarrow \frac{1}{2} du = (x + 1)dx$$

3) If that doesn't work, you may need to complete the square (or do partial fractions)

$$\int \frac{1}{u} du = \ln u + C = \ln|x^2 + 2x + 6| + C$$

Example.

$$\int \frac{3}{e^x + 1} dx = \int \frac{3}{e^x + 1} \times \left(\frac{e^{-x}}{e^{-x}}\right) dx = \int \frac{e^{-x}}{1 + e^{-x}} dx$$

$$u = 1 + e^{-x}, du = -e^{-x} dx \rightarrow -du = e^{-x} dx$$

$$\int -\frac{1}{u} du \; = -\ln u + C = -\ln|e^{-x} + 1| + C$$

Example.

$$\int \frac{\sin x - \cos x}{\sin x + \cos x} dx$$

$$u = \sin x + \cos x\,, du = (\cos x - \sin x) dx \rightarrow -du = (\sin x - \cos x) dx$$

$$\int -\frac{1}{u} du = -\ln|\sin x + \cos x| + C$$

Alternative way

You can also consider ways of converting by multiplication to get $\cos^2 x + \sin^2 x = 1$, or $\cos^2 x - \sin^2 x = cos\, 2x$

Inverse trig functions

$$\int \frac{1}{\sqrt{1 - x^2}} dx = \arcsin x + C$$

$$\int \frac{1}{1 + x^2} dx = \arctan x + C$$

$$\int \frac{1}{x\sqrt{x^2 - 1}} dx = \operatorname{arcsec} x + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \operatorname{arcsec} \frac{x}{a} + C$$

U-sub	Inverse Trig
$\int \frac{x}{\sqrt{1-x^2}}dx, u = 1 - x^2$	$\int \frac{1}{\sqrt{1-x^2}}dx$
$\int \frac{x}{1+x^2}dx\,, u = 1 + x^2$	$\int \frac{1}{1+x^2}dx$
$\int \frac{x}{\sqrt{x^2-1}}dx$	$\int \frac{1}{x\sqrt{x^2-1}}dx$

The sign between the terms doesn't matter in u-sub, but does in inverse trig.

Example.

$$\int \frac{3}{4+x^2}dx = \frac{3}{2}\arctan\frac{x}{2} + C$$

Example.

$$\int \frac{\cos x}{2+\sin^2 x}dx =$$
$$u = \sin x\,, du = \cos x\, dx$$

$$\int \frac{1}{2+u^2}du = \frac{1}{\sqrt{2}}\arctan\frac{u}{\sqrt{2}} + C = \frac{1}{\sqrt{2}}\arctan\frac{\sin x}{\sqrt{2}} + C$$

Example.

$$\int \frac{dx}{x\sqrt{4x^2-25}} = \int \frac{2dx}{2x\sqrt{4x^2-25}}$$

$$u = 2x, du = 2dx$$

$$\int \frac{du}{u\sqrt{u^2-25}} = \frac{1}{5}\text{arcsec}\frac{u}{5} + C = \frac{1}{5}\text{arcsec}\frac{2x}{5} + C$$

Example.

$$\int \frac{\arccos x}{\sqrt{1-x^2}}dx =$$
$$u = \arccos x\,, du = -\frac{1}{\sqrt{1-x^2}}dx \to -du = \frac{1}{\sqrt{1-x^2}}dx$$

$$\int -udu = -\frac{1}{2}u^2 + C = -\arccos^2 x + C$$

Example.

$$\int \frac{e^x}{9+e^{2x}}dx$$

$$u = e^x, du = e^x dx$$

$$\int \frac{du}{9+u^2} = \frac{1}{3}\arctan\frac{u}{3} + C = \frac{1}{3}\arctan\frac{e^x}{3} + C$$

Example.

$$\int \frac{1}{(2x+2)\sqrt{x}}dx = \int \frac{1}{2(x+1)\sqrt{x}}dx$$

$$u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}dx, u^2 = x$$

$$\int \frac{1}{u^2+1}\, du = \arctan u + C = \arctan \sqrt{x} + C$$

If arcsine is involved:

$$\int \frac{1}{\sqrt{x}\sqrt{1-x}}dx$$

Arcsecant:

$$\int \frac{1}{x\sqrt{x-1}}dx = \int \frac{1}{\sqrt{x}\sqrt{x}\sqrt{x-1}}dx$$

These are the hardest to recognize that they are arcsecants, but we don't have another method to use either.

Example.

$$\int \frac{1}{x^2+2x+5}dx = \int \frac{1}{x^2+2x+1+4}dx = \int \frac{1}{(x+1)^2+4}dx$$

$$u = x+1, du = dx$$

$$\int \frac{1}{u^2+4}dx = \frac{1}{2}\arctan\frac{u}{2} + C = \frac{1}{2}\arctan\frac{x+1}{2} + C$$

Hyperbolic trig functions

Regular Trig	Hyperbolic Trig
$\int \sin x\, dx = -\cos x\, dx + C$	$\int \sinh x\, dx = \cosh x\, dx + C$
$\int \cos x\, dx = \sin x\, dx + C$	$\int \cosh x\, dx = \sinh x\, dx + C$
$\int \sec^2 x\, dx = \tan x + C$	$\int \text{sech}^2\, x\, dx = \tanh x + C$
$\int \sec x \tan x\, dx = \sec x + C$	$\int \text{sech}\, x \tanh x\, dx = -\text{sech}\, x + C$
$\int csc^2\, x\, dx = -\cot x + C$	$\int csch^2\, x\, dx = -\coth x + C$
$\int \csc x \cot x\, dx = -\csc x + C$	$\int \text{csch}\, x \coth x\, dx = -\text{csch}\, x + C$

$$\int \frac{\cosh x}{(1 + \sinh x)} dx$$

$$u = 1 + \sinh x \,, du = \cosh x \, dx$$

$$\int \frac{1}{u} du = \ln u + C = \ln |1 + \sinh x\,| + C$$

$$\int \frac{\cosh x}{(1 + \sinh^2 x)} dx$$

$$u = \sinh x \,, du = \cosh x \, dx$$

$$\int \frac{1}{1 + u^2} du = \arctan u + C = \arctan(\sinh x) + C$$

We will not worry about the inverse hyperbolic trig functions.

On Thursday, we will review for Exam #2. Bring your questions. We will go until you run out of questions (so possibly be a short lecture?).