8/1/2023

Arc Length and Surfaces of Revolution
Work and Probability
Centers of Mass

Arc Length
Measuring the length of a curve as you follow along it.
Formula

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

Numerical integration in your calculator:
MATH → 9 fnInt(
fnInt(function, x, lower limit, upper limit)

Functions that work well:

$$y = \ln(\cos x)$$
$$y = \cosh(x)$$
$$y = x^{\frac{3}{2}}$$

A class of functions that combine a polynomial term and rational term (negative exponent), where if you take the derivative y' has a x^n and x^{-n} power, and it simplifies with the +1 and can then be refactored as a perfect square.

Example.
Find the length of the curve $y = \ln(\cos x)$ between $x = 0, x = \frac{\pi}{4}$.

$$y' = \frac{1}{\cos x}(-\sin x) = -\tan x$$

$$s = \int_0^{\frac{\pi}{4}} \sqrt{1 + (-\tan x)^2} dx = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x}\, dx = \int_0^{\frac{\pi}{4}} \sec x\, dx = \ln|\sec x + \tan x|_0^{\frac{\pi}{4}}$$

$$= \ln\left|\sqrt{2} + 1\right|$$

Example.
Find the length of the curve $y = \frac{x^3}{3} + \frac{1}{4x} = \frac{1}{3}x^3 + \frac{1}{4}x^{-1}$ from $x = 1$ to $x = 3$

$$y' = x^2 - \frac{1}{4}x^{-2}$$

$$s = \int_1^3 \sqrt{1 + \left(x^2 - \frac{1}{4}x^{-2}\right)^2}\, dx = \int_1^3 \sqrt{1 + x^4 - \frac{1}{4} - \frac{1}{4} + \frac{1}{16}x^{-4} dx}$$

$$= \int_1^3 \sqrt{1 + x^4 - \frac{1}{2} + \frac{1}{16}x^{-4}}dx = \int_1^3 \sqrt{x^4 + \frac{1}{2} + \frac{1}{16}x^{-4}}dx = \int_1^3 \sqrt{\left(x^2 + \frac{1}{4}x^{-2}\right)^2}\, dx =$$

$$\int_1^3 \left(x^2 + \frac{1}{4}x^{-2}\right) dx = \frac{x^3}{3} - \frac{1}{4x}\bigg|_1^3 = 9 - \frac{1}{12} - \frac{1}{3} + \frac{1}{4} = \frac{53}{6}$$

Example.
Find the arc length of $f(x) = x^2$ between $x = 0$ and $x = 4$.

$$f'(x) = 2x$$

$$s = \int_0^4 \sqrt{1 + (2x)^2}dx = \int_0^4 \sqrt{1 + 4x^2}dx$$

Integrating this expression requires trig substitution, which is next semester.

Surface area for a solid of revolution.

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}dx$$

$f(x)$ described the curve that is being rotated. $r(x)$ will change depending on which axis we are rotating around.

If you are rotating around the x-axis, the radius $r(x) = f(x)$.
If you are rotating around the y-axis, the radius $r(x) = x$

Example.
Find the surface area of revolution for rotating $y = x^2$ around the y-axis between $x = 0, x = 2$.

$$S = 2\pi \int_0^2 x\sqrt{1 + 4x^2}dx$$

Let $u = 1 + 4x^2, du = 8xdx, \frac{1}{8}du = xdx \ \rightarrow \int \frac{1}{8}u^{\frac{1}{2}}du$

$$2\pi \left[\frac{1}{8}(1 + 4x^2)^{\frac{3}{2}}\left(\frac{2}{3}\right)\right]_0^2 = \frac{1}{2}\left(\frac{1}{3}\right)\pi\left[17^{\frac{3}{2}} - 1\right]$$

Example.
Find the surface area of revolution for rotating $y = x^2$ around the x-axis between $x = 0, x = 2$.

$$S = 2\pi \int_0^2 x^2\sqrt{1 + 4x^2}dx$$

Now I need either integration by parts, or change of variables (?), or trig substitution…

Probability applications

Continuous probability distribution has the area under the curve representing the probability.
The area under the curve must be equal 1 to be valid probability distribution. We can validate probability distribution using calculus, or find the constant that we need to multiply the function by to make it a valid probability distribution.

Consider the function $f(x) = k(1 - x^3)$ on the interval [0,1]. Find the value of k to make this a valid probability distribution.

$$1 = k\int_0^1 1 - x^3 dx = k\left[x - \frac{1}{4}x^4\right]_0^1 = k\left[1 - \frac{1}{4}\right] = k\left(\frac{3}{4}\right)$$

$$k = \frac{4}{3}$$

Probability distribution is $f(x) = \frac{4}{3}(1 - x^3)$.

I can calculate probabilities with it. Continuous only have probabilities on intervals, not a single value.

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Find the probability that X is less than 0.5. Find $P(X < 0.5)$.

$$P(X < 0.5) = \int_0^{0.5} \frac{4}{3}(1 - x^3)dx = \frac{4}{3}\left[x - \frac{1}{4}x^4\right]_0^{0.5} = \frac{4}{3}\left[\frac{1}{2} - \frac{1}{4}\left(\frac{1}{16}\right)\right] = \frac{31}{48} \approx 0.6458$$

Finding the median, mean and variance using calculus.

To find the median:

$$0.5 = \int_0^x \frac{4}{3}(1 - t^3)dt$$

Solve for x numerically in your calculator. (using the zero function)

To find the mean:

$$E(X) = \mu = \int_a^b xf(x)dx = \int_0^1 \frac{4}{3}x(1 - x^3)dx = \frac{2}{5}$$

Multiply the distribution function by x and then integrate over the whole interval.

The variance is

$$V(X) = \sigma^2 = \int_a^b (x - \mu)^2 f(x)dx = \int_0^1 \frac{4}{3}(x - \mu)^2(1 - x^3)dx = \int_0^1 \frac{4}{3}\left(x - \frac{2}{5}\right)^2 (1 - x^3)dx = \frac{14}{225}$$

(the standard deviation is the square root of this expression).

Work

When things are constant, we have $W = Fd$, but when either the force is changing or the distance is different, then we need calculus and integration.

$$W = \int_a^b \frac{dF}{dx} D(x)dx = \int_a^b F(x)dx$$

Springs

Hooke's Law $F = kx$

Find the work done by a spring. It takes 10 pounds to compress a spring 2 inches. Find the work done in compressing the spring an additional 4 inches.

Gravity to convert mass to force (weight) is 9.8 m/s^2 in SI, or 32 ft/s^2 in Imperial units.

$$10 = k\left(\frac{1}{6}\right)$$
$$k = 60$$

$$W = \int_{\frac{1}{6}}^{\frac{1}{4}} 60x dx = 30x^2 |_{\frac{1}{6}}^{\frac{1}{4}} = 30\left(\frac{1}{4}^2 - \frac{1}{6}^2\right) = \frac{25}{24} \ ft - lbs$$

Gravity problems

$$F = \frac{Gm_1m_2}{d^2} = \frac{k}{x^2}$$

Gravity is measured from the center of the earth (or the moon… of planetary object you are standing on). A weight is measured on the surface, so they will have to give you an approximation for the radius of the object.

Suppose you want to launch a 20-ton satellite into orbit 1000 miles above the surface of the earth. Assume the radius of the earth is 4000 miles.

$$W = \int_{4000}^{5000} \frac{k}{x^2} dx$$

Chains

There is an example in the handout. Set up the density of a small section of the chain, and then how far you have to move that section of the chain. Integrate to get the work.

Tank

The hard part about the tank problems is that they differ based on the geometry of the tank.

Find the work done in pumping the water out of a cylindrical tank that is 10 feet high and has a radius of 6 feet, if the tank is full.

Think about a cross-sectional slice of the tank. The radius of each cross-section is 6 feet. And therefore if we imagine a cylindrical slice of the tank: the surface of that cylindrical slice is $\pi r^2 = 36\pi$. The volume is the surface area (circle) times the width… dx.

Volume is $36\pi dx$.
Weight is density of water times the volume:
62.4 lbs/ft^3 in imperial or 9800 N/m^3

Distance to move the slice: $height\ of\ the\ tank - distance\ from\ the\ bottom\ of\ the\ tank$

$$10 - x$$

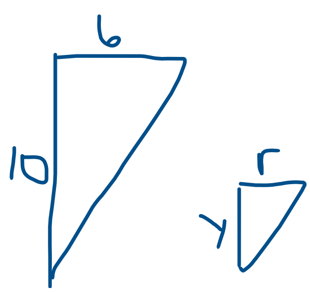
$$W = \int_0^{10} 62.4(36\pi)(10-x)dx = 62.4(36\pi)\int_0^{10} 10 - x\,dx = 62.4(36\pi)\left[10x - \frac{1}{2}x^2\right]_0^{10} =$$

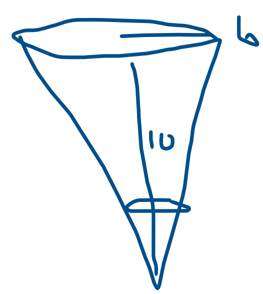
$$62.4(36\pi)(100-50) = 112{,}320\pi\ ft - lbs$$

Conical Tank.

We have a conical tank that has a height of 10 feet and a radius at the top of 6 feet. (point is downward) Find the work done in emptying the tank full of water.

Area of the cross section is related to πr^2, but the radius is changing with the height of the tank. To come up with an equation that relates the radius to the height we use similar triangles.





$$\frac{6}{10} = \frac{r}{y}$$

$$r = \frac{3}{5}y$$

Area: $\pi\left(\frac{3}{5}y\right)^2 = \frac{9\pi}{25}y^2$
Volume: $\frac{9\pi}{25}y^2dy$
Weight: $62.4\left(\frac{9\pi}{25}y^2dy\right)$

Distance being moved: height – y = 10-y

$$W = \int_0^{10} 62.4\left(\frac{9\pi}{25}y^2\right)(10-y)dy = (62.4)\frac{9\pi}{25}\int_0^{10} 10y^2 - y^3 dy =$$

$$(62.4)\frac{9\pi}{25}\left[\frac{10}{3}y^3 - \frac{1}{4}y^4\right]_0^{10} = (62.4)\frac{9\pi}{25}\left[\frac{10000}{3} - \frac{10000}{4}\right] = 18720\pi$$

Centers of Mass
On lamina (thin sheet) with constant density
Typically the region is bounded by one function and an axis, or by two functions

$$M = \rho\int_a^b [f(x) - g(x)]dx$$

This is the total mass.

Moment of mass from the x-axis:

$$M_x = \frac{\rho}{2}\int_a^b [f(x)^2 - g(x)^2]dx$$

Moment of mass from the y-axis:

$$M_y = \rho\int_a^b x[f(x) - g(x)]dx$$

Center of mass: $(\bar{x}, \bar{y})$, where $\bar{x} = \frac{M_y}{M}$, and $\bar{y} = \frac{M_x}{M}$

Find the center of mass of the lamina bounded by $y = x^2, y = x$

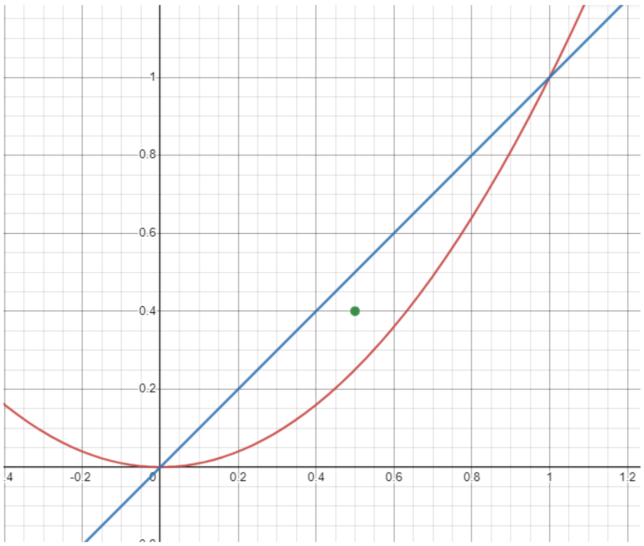
$$M = \int_0^1 x - x^2 dx = \frac{1}{2}x - \frac{1}{3}x^3\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$M_x = \frac{1}{2}\int_0^1 x^2 - x^4 dx = \frac{1}{2}\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \left(\frac{1}{2}\right)\left(\frac{2}{15}\right) = \frac{1}{15}$$

$$M_y = \int_0^1 x(x - x^2)dx = \int_0^1 x^2 - x^3 dx = \frac{1}{3}x^3 - \frac{1}{4}x^4\Big|_0^1 = \frac{1}{12}$$

$$\bar{x} = \frac{\left(\frac{1}{12}\right)}{\frac{1}{6}} = \frac{1}{12}\times\frac{6}{1} = \frac{1}{2}$$

$$\bar{y} = \frac{\frac{1}{15}}{\frac{1}{6}} = \frac{1}{15}\times\left(\frac{6}{1}\right) = \frac{2}{5}$$



Center $\left(\frac{1}{2}, \frac{2}{5}\right)$